ECONOMETRICA

JOURNAL OF THE ECONOMETRIC SOCIETY

An International Society for the Advancement of Economic Theory in its Relation to Statistics and Mathematics

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VOL. 6, NO. 2 — APRIL, 1938

ECONOMETRICA

Published quarterly by the Econometric Society
Colorado Springs, Colorado, U. S. A.

BUSINESS OFFICE OF ECONOMETRICA, Mining Exchange Building, Colorado Springs, U. S. A. In the absence of specifications to the contrary, any communication regarding ECONOMETRICA or the Econometric Society, may be addressed to this office.

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Subscription Price $7.00 per annum. Single copies $1.75.

Printed in the U. S. A. by George Banta Publishing Company
Menasha, Wisconsin

ECONOMETRICA

VOLUME 6 APRIL, 1938 NUMBER 2

THE DETERMINANTS OF DISTRIBUTION OF THE NATIONAL INCOME

BY M. KALECKI

IN THE PRESENT PAPER we try to investigate both statistically and analytically the problem of the relative share of manual labour in the national income. There are two reasons why we do not consider the total share of labour, although it would be more interesting from the social point of view: (1) The statistics of national income include in the salaries the incomes of directors, managers, etc., which should rather be placed under the heading of profits. In this way what statistics give as the total share of labour does not represent correctly the distribution of the product of industry between profits and interest on the one hand, and wages and salaries on the other. (2) The relative share of *manual labour* in the national income is more suitable for theoretical analysis.

It must be added that we shall deal here with the relative share of manual labour not in net but in *gross* income, by which is meant the income before deduction of maintenance and depreciation. (Gross income = net income + maintenance and depreciation.) The reasons for this are again both "technical" and theoretical: (1) The statistics of gross income are much more reliable than those of net income; the latter are based on the net incomes of firms whose allowance for depreciation[1] is certainly more or less arbitrary. In adding to the net national income aggregate depreciation, as given by the firms' accounting, we obtain gross income free from this arbitrariness. (2) It is the share of manual labour in the *gross* national income which—as we shall see below—*ex natura rei* can be more easily treated theoretically.

THE STATISTICAL DATA

1. The figures for Great Britain are based on Professor Bowley's *The Change in the Distribution of the National Income, 1880–1913*, and Mr. Colin Clark's *National Income and Outlay*.

Using Professor Bowley's data on the distribution of national income (p. 16) and deducting from total income the interest from overseas

[1] For the sake of brevity we shall speak throughout the paper of "depreciation" instead of "maintenance and depreciation."

(mentioned on p. 25) we obtain the relative share of manual labour in home-produced income: 43.5 in 1880, and 39.3 in 1913. (It is the distribution of home-produced income in which we are interested.) The above figures are shares in net income—Professor Bowley does not give data on depreciation and gross income. The rate of increase of gross income in that period, however, is unlikely to differ much from that of net income: the proportion of depreciation to net income in 1913 was only about 8 per cent and the volume of capital equipment and national income in 1913 and 1880 show that this percentage could not have undergone a great change within this period.[2] Thus the relative share of manual labour in gross income altered within the period in question similarly to that in net income.

The figures for 1911 and 1924–35 are obtained on the basis of Mr. Colin Clark's data on "Distribution of Income between Factors of Production, 1911 and 1924–35" (*National Income and Outlay*. p. 94) and depreciation (pp. 86, 169). The relative shares here calculated differ from those given by Mr. Clark (p. 94) only in that they are taken in relation to *gross* home-produced income.

TABLE 1
RELATIVE SHARE OF MANUAL LABOUR IN THE NATIONAL INCOME

	1880	1911	1913	1924	1925	1926	1927	1928
In net income (Bowley)	43.5		39.3					
In gross income (Clark)		36.5		38.2	37.5	37.0	38.3	38.2

	1929	1930	1931	1932	1933	1934	1935	
In net income (Bowley)								
In gross income (Clark)	37.3	36.5	38.6	38.0	37.5	37.3	36.7	

We see that the relative share of manual labour in the national income in Great Britain declined moderately between 1880 and 1913 and showed a remarkable stability between 1913 and 1935 both in the long run and in the short period.

2. The figures for the U. S. A. are based on Dr. King's *The National Income and Its Purchasing Power, 1909–1928* and a recent estimate of national income and depreciation by Dr. Kuznets.

[2] The real capital per head increased by about 25 per cent, the real income per head by about 40 per cent (*National Income and Outlay*, pp. 273 and 232) while the rate of depreciation was probably to some extent higher in 1913 than in 1880.

The relative shares of manual labour in the net national income in 1909 and 1928 are according to King 33.7 and 32.4.[3] Also here the change of relative shares in gross income is probably not very different.

For the period 1929–35 (most interesting from the standpoint of short-period analysis), Dr. Kuznets' estimates are used. These estimates, however, give separate figures of wages and salaries only for "selected industries" (manufacturing, mining, construction, and transport). Thus here we could only calculate the share of manual labour in the income of this part of United States economy. But in spite of this the figures obtained are quite valuable for our investigation. For a section of a national economy can of course be treated as an open economic system, and in our theoretical analysis we do not suppose the system to be closed. (Otherwise the figures concerning the English national income could not be taken into consideration either.)

We obtained the gross income of "selected industries" by adding to their "income produced"[4] the depreciation estimated by Dr. Kuznets in *Gross Capital Formation.*[5] The relative shares of manual labour in this gross income are given in the following table:

TABLE 2

	1909	1928	1929	1930	1931	1932	1933	1934	1935
In net national income (King)	33.7	32.4							
In gross income of manufacturing, mining, construction, and transport (Kuznets)			40.0	42.2	42.0	41.0	37.8	39.5	39.5

Here, too, the long-run change between 1909 and 1928 is very small. Fluctuations in the period 1929–35, however, are much greater than in Great Britain, no doubt owing to the violent disturbances in the

[3] *The National Income*, p. 80. We have excluded from income the services of durable consumption goods which King treats as a part of national income (he calls this part "imputed income"). We have also excluded from King's figures of wage income that of shop assistants, which we treat throughout (according to Clark) as salaries.

[4] *Survey of Current Business*, May, 1936.

[5] Pp. 11 and 12. Depreciation and maintenance is estimated here for the whole economy (dwelling houses excluded) but as a matter of fact it can be almost totally attributed to manufacturing, mining, construction, and transport. Mr. Kuznets has afterwards corrected these figures and we introduce these corrections as given in Mr. Keynes' note, *Economic Journal*, September, 1936. For 1934 and 1935 we were obliged to make our own rough estimates by means of interpolation.

United States economy while the depression in Great Britain was relatively mild. With the exception of 1933, however, the difference from the average is not great.

As we see on the basis of statistical data the relative share of manual labour in gross income shows only small changes both in the long run and in the short period. We shall try to explain this "law" and establish conditions under which it is valid.

THE DEGREE OF MONOPOLY AND THE DISTRIBUTION OF THE PRODUCT OF INDUSTRY

Let us consider an enterprise with a given capital equipment which produces at a given moment an output x and sells it at price p. The short-period marginal cost m (i.e., the cost of producing an additional unit of product with a given capital equipment) is made up of the sum of the short-period marginal costs of: depreciation d_m (caused by greater use of equipment), salaries s_m, wages w_m, and raw materials r_m:

$$m = d_m + s_m + w_m + r_m.$$

At the same time the price is equal to the sum of the corresponding average costs d_a, s_a, w_a, r_a and the average capitalist income (profit and interest) c_a per unit of output:

$$p = c_a + d_a + s_a + w_a + r_a.$$ [6]

We substract the first equation from the second and obtain:

$$(1) \quad p - m = c_a + (d_a - d_m) + (s_a - s_m) + (w_a - w_m) + (r_a - r_m).$$

According to Mr. Lerner[7] we shall call the degree of monopoly of the enterprise μ, the ratio of the difference between price and marginal cost to price, or:

$$\mu = \frac{p - m}{p}.$$

If marginal cost is equal to marginal revenue, μ is equal to the inverse of the elasticity of demand for the product of the enterprise. Substituting $p\mu$ for $p-m$ in the equation (1) and multiplying both sides by the output x we get:

$$xp\mu = xc_a + x(d_a - d_m) + x(s_a - s_m) + x(w_a - w_m) + x(r_a - r_m).$$

Such an equation can be written for each enterprise of an economy (or

[6] We mean here by p the "net price," i.e., the revenue per unit of product after deduction of advertising costs, etc.

[7] "The Concept of Monopoly and the Measurement of Monopoly Power," *Review of Economic Studies*, Vol. 1, June, 1934, pp. 157–175.

any of its sections). By adding the equations for all enterprises we obtain:

$$\sum xp\mu = \sum xc_a + \sum x(d_a - d_m) + \sum x(s_a - s_m) + \sum x(w_a - w_m) + \sum x(r_a - r_m). \tag{2}$$

The sum $\sum xc_a$ is nothing else than the aggregate net capitalist income C. Further, in a great majority of enterprises marginal depreciation d_m is small in comparison with average depreciation d_a; thus $\sum x(d_a - d_m)$ can be represented by $D(1-\alpha)$ where D is the aggregate depreciation and α a small positive fraction. For similar reasons $\sum x(s_a - s_m)$ can be represented by $S(1-\beta)$ where S is the aggregate salary and β a small positive fraction.

We are now going to examine the member $\sum x(w_a - w_m)$. The shape of the average wage-cost curve differs in various types of enterprise. In most industries the average cost of manual labour falls slowly, remains constant, or rises slowly until full employment in two shifts of six days a week is reached. In the second type of enterprise the average wage cost falls rather sharply before this point is attained (railways). In the third type a sharp rise of average wage cost begins relatively early (agriculture and mining). If we consider now an economy in which: (1) The second and third type do not produce a large percentage of turnover; (2) Most enterprises of the first two types do not exceed full employment in two shifts six days a week; then the sum $\sum x(w_a - w_m)$ is likely to be small in comparison with the aggregate wage W. Indeed, the greatest part of income is produced under conditions of slowly changing manual-labour cost, and thus by enterprises for which $w_a - w_m$ is small in relation to W. For the second and third type this difference is not so small in comparison with W, and positive or negative respectively. Since neither of the latter types produces a large proportion of aggregate turnover $\sum px$, it will be easily seen that the sum $\sum x(w_a - w_m)$ is probably small in comparison with W. It can thus be represented by γW where γ is a small positive or negative fraction.

As concerns the average cost of raw materials it can be supposed approximately constant, and consequently the sum $\sum x(r_a - r_m)$ can be neglected as being near to O.

On the basis of the above assumptions which seem to hold good for highly developed industrial economies we can write equation (2) as follows:

$$\sum xp\mu = C + D(1 - \alpha) + S(1 - \beta) + \gamma W$$

or

$$\sum xp\mu = (C + D + S) - (D\alpha + S\beta - \gamma W)$$

where α, β, and γ are small fractions.

Now it is obvious that both αD and βS are small in relation to $C+D+S$; but the same can be said about γW since, as the statistical data quoted above show, W is less than half of the gross income Y and thus less than $Y-W=C+D+S$. We can consequently conclude that $\alpha D+\beta S-\gamma W$ is also small in comparison with $C+D+S$, and therefore:

$$\sum xp\mu = C + D + S$$

can be regarded as a good approximation. Now let us divide both sides of this equation by the aggregate turnover $T=\sum xp$.

$$\frac{\sum xp\mu}{\sum xp} = \frac{C + D + S}{T}.$$

The expression on the left-hand side of this equation is nothing else than the weighted average of the degree of monopoly μ which we shall denote by (μ). We have thus the following proposition: *The relative share of gross capitalist income and salaries in the aggregate turnover is with great approximation equal to the average degree of monopoly:*

$$\frac{C + D + S}{T} = (\mu).\text{[8]} \tag{3}$$

HOW IS IT POSSIBLE FOR THE DEGREE OF MONOPOLY TO DETERMINE THE DISTRIBUTION OF THE PRODUCT OF INDUSTRY?

1. The results attained in the last section may indeed seem paradoxical. In the case of free competition the average degree of monopoly (μ) is equal to zero; thus equation (3) seems to show that free competition makes it impossible not only to earn profits and interest, but even to cover depreciation and expenses for salaries—all gross income being absorbed by wages. This paradox is, however, only apparent. The formula (3) can be correct merely when the assumptions on which is is based are fulfilled. According to these assumptions: (1) The short-period marginal-cost curve does not differ considerably in the majority of enterprises from the short-period average-cost curve of manual labour and raw materials—up to a certain point (where full employment of the factory in two shifts of six days a week is reached). (2) The output in these enterprises is mostly below this point. These assumptions are quite realistic, but such a state of affairs is possible only with the existence of monopoly or imperfect competition. If free competition prevails, the second condition cannot be fulfilled: enterprises must close

[8] This formula will also be valid for a section of a national economy if the basic assumptions are fulfilled. This will clearly be the case for the set of "selected industries" in U. S. A. (see pp. 97–100) including manufacturing, mining, construction, and transport.

down or maintain such a degree of employment that the marginal cost is higher than the average cost of manual labour and raw materials. In the real world an enterprise is seldom fully employed in two shifts of six days a week, a fact which is therefore a demonstration of market imperfection and widespread monopolies. And our formula, though quite realistic, is not applicable in the case of free competition.

The second question which may be raised is of a more complex character. Since, according to our formula, the distribution of the product is at every moment determined by the degree of monopoly, it therefore holds both for the short period and in the long run. The formula was, however, deduced on the basis of, so to speak, pure short-period considerations. And both the elasticity of substitution between capital and labour, and inventions are, contrary to the prevailing opinion, of no influence on the distribution of income.

The answer is: (1) That the long-period analysis of distribution is generally conducted on a basis of oversimplified representation of output as a function of only two variables—capital (taken *in abstracto*) and labour. In this way, the short-period cost curves are—as we shall see at once—excluded artificially from this analysis. (2) On the basis of our assumptions these curves have a special shape which makes for the elimination of factors other than the degree of monopoly from the mechanism of distribution. To clarify the problems concerned we shall now consider the dependence of long-run distribution of the product of industry on the shape of the short-period cost curves.

2. A particular commodity can be produced with various types of equipment requiring more or less labour and raw materials per unit of product. The conditions of production are, however, determined not only by the choice of the type of equipment but also by its use. Not only may the kind of machinery be varied but it is also, for example, possible to work with the same machinery in one or two shifts.

Let us assume for a moment free competition and draw for each alternative type of equipment, which can be applied in the production of the commodity considered, a short-period marginal-cost curve and a short-period average-cost curve of manual labour and raw materials (Figure 1). The shaded area then represents the value of net capitalist income, depreciation, and salaries, while the unshaded area LMNO represents the cost of manual labour and raw materials.

To determine the position of long-period equilibrium we define first for each type of equipment the level of prices at which the shaded area covers salaries, depreciation, interest, and normal profit (i.e., the rate of profit at which the industry in question neither expands nor contracts). We shall call this price the normal price attached to a given type of equipment, and the corresponding use of this equipment, its

normal use.[9] From all types of equipment we choose that to which the lowest normal price is attached. It is easy to see that the normal use of this type of equipment represents the long-run equilibrium. It is clear now that the shape of the short-period marginal-cost curves corresponding to various types of equipment influences the formation of long-run equilibrium.

If some change in basic data takes place—e.g., the rate of interest alters or new invention occurs, the long-run equilibrium is shifted; a new type of equipment is used in a "normal" way, and in general the relation of shaded and unshaded areas will be different from that in the

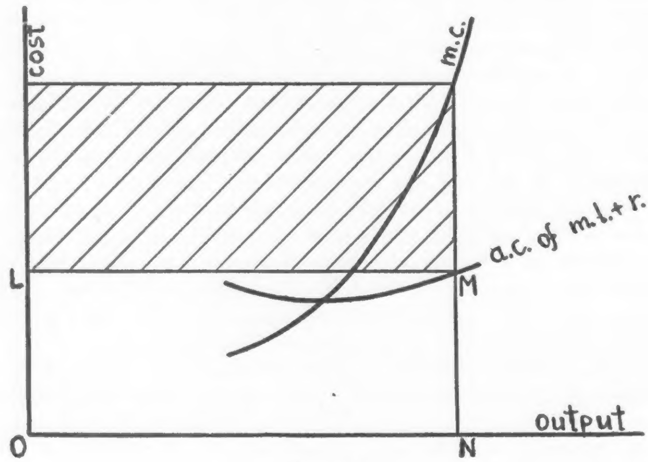


FIGURE 1

initial position. This is quite in accordance with the prevailing long-run theory of distribution. We shall see, however, that such is not the case with the peculiar shape of marginal-cost curves assumed in the deduction of formula (3) and if we admit, instead of free competition, a certain given degree of monopoly.

We take for granted that the short-period marginal-cost curve does not differ appreciably from the average-cost curve of manual labour and raw materials, below the point A (Figure 2). We represent them thus by the same thick curve PMB. With a given degree of monopoly the relation of price and marginal cost is a constant $1/(1-\mu)$. Thus if output remains below OA the price corresponding to it is represented

[9] It is easy to see that with free competition the normal use coincides with so-called optimum use.

by the curve QRC, whose ordinates are proportionate to those of the curve PMB. The ratio of shaded areas representing profits, interest, depreciation, and salaries to the unshaded area representing wages and cost of raw materials is equal to $\mu/(1-\mu)$.

We define in exactly the same way as before the normal use for each type of equipment as that at which normal profit is earned. The long-run equilibrium is again represented by the normal use of such type of equipment that—with a given degree of monopoly—it is impossible to earn profits higher than normal in employing plants of a different type. If the basic data alter, the new long-run equilibrium is repre-

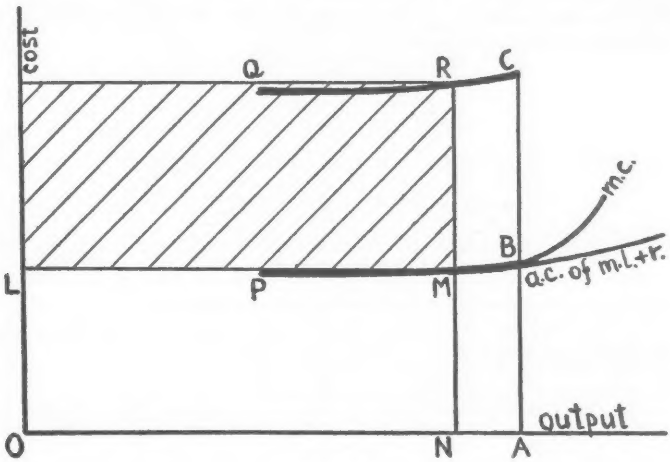


FIGURE 2

sented by the normal use of a different type of equipment. The long-run equilibrium price of the product alters too, but not its relation to the average cost of manual labour and raw material, since for all types of equipment the marginal-cost curve coincides with the average-cost curve of manual labour, and the degree of monopoly (which is equal to the relation of price to marginal cost) is supposed to be given. In that way the distribution of the product among factors, as expressed by the relation of shaded and unshaded area, remains unaffected by changes of basic data so long as the degree of monopoly is unaltered and the use of equipment in long-run equilibrium does not reach the point A.

The change of basic data may of course also influence the degree of monopoly. For instance, a change in the rate of interest or technical

progress affects the size of the enterprise which is essential for the degree of monopoly. (The variation of the scale can be treated as a special case of variation of the type of equipment.) In that way such changes influence the distribution of income, but this is not in contradiction with our results because it is the channel of the degree of monopoly through which this influence makes itself felt.

DISTRIBUTION OF THE NATIONAL INCOME

1. Our proper task is to find the relative share of wages W in the national income Y. Since Y is equal to the sum of capitalists' income C, depreciation D, salaries S, and wages W, it amounts to the same as to find the determinants of $(C+D+S)/Y=(Y-W)/Y$. In multiplying each side of the equation

$$\frac{C+D+S}{T}=(\mu)$$

by the ratio of turnover T and gross income Y we obtain:

$$\frac{C+D+S}{Y}=(\mu)\frac{T}{Y}\cdot$$

Thus it is the degree of monopoly (μ) and the ratio of turnover to income which determines the distribution of income.

On the basis of Mr. Colin Clark's data[10] there was in the manufacturing industries of Great Britain in 1934:

$$(\mu)=0.23;\frac{T}{Y}=2.3;\frac{C+D+S}{Y}=0.23\times 2.3=5.3.$$

I ventured to make a rough estimate (based also on Mr. Clark's data) for the entire economy of Great Britain and obtained:

$$(\mu)=0.30;\frac{T}{Y}=2.1;\frac{C+D+S}{Y}=0.30\times 2.1=0.63.$$

2. The factors (μ) and T/Y are not independent: a change in the degree of monopoly influences the ratio of turnover to income. As we shall see on the basis of a numerical example a rise (fall) of the degree of monopoly causes a decrease (increase) of T/Y but in a lesser proportion.

We shall consider for our purpose an economy producing its output in two stages: in the first semimanufactured goods are produced from foreign raw materials; in the second the former are worked up into

[10] *National Income and Outlay*, pp. 132, 133.

finished commodities. Let the numerical scheme of production be as follows:

	Raw materials	Wages	Profits, interest, depreciation, salaries	Turnover
1st stage	4	2	2	8
2nd stage	8	4	4	16
Item	12	6	6	24

The degree of monopoly is:

$$(\mu) = \frac{C + D + S}{T} = \frac{6}{24} = 0.25,$$

and the relation of aggregate turnover to aggregate income:

$$\frac{T}{Y} = \frac{T}{W + (C + D + S)} = \frac{24}{6 + 6} = 2.0.$$

Let us now suppose that the degree of monopoly has changed at all stages by 10 per cent. Our scheme will then alter as follows:

	Raw materials	Wages	Profits, interest, depreciation, salaries	Turnover
1st stage	4	2	2.27	8.27
2nd stage	8.27	4	4.65	16.92
Item	12.27	6.00	6.92	25.19

The degree of monopoly is now (in accordance with the assumption of a 10 per cent change):

$$(\mu) = \frac{6.92}{25.19} = 0.275,$$

and the ratio of turnover to income:

$$\frac{T}{Y} = \frac{25.19}{6.00 + 6.92} = 1.95.$$

Thus the ratio of turnover has diminished by 2.5 per cent as a result of the increase in the degree of monopoly by 10 per cent.

We can now conclude that the relative share $(C+D+S)/Y$ increases *caeteris paribus* with the rise of the degree of monopoly but in a lesser

proportion than the latter. [In the example above $(C+D+S)/Y$ increased from $6/(6+6)=0.5$ to $6.92/(6.00+6.92)=0.536$, i.e., by 7.2 per cent as a result of the 10 per cent increase of (μ).]

3. Changes in T/Y can, of course, be caused by influences other than a change in (μ). A change in the price of "basic raw materials"—i.e., of products of agriculture and mining in relation to wages in other industries—will clearly also have an important influence. It is easy to see on the basis of our scheme that the ratio of turnover to income increases (decreases) with the rise (fall) of the prices of basic raw materials in relation to wages, but in a much smaller proportion.

Let us suppose that the raw materials manufactured in the first stage (we assumed them to be of foreign origin) become 50 per cent dearer, wages (and the degree of monopoly) remaining unaltered. Then our scheme alters as follows:

	Raw materials	Wages	Profits, interest, depreciation, salaries	Turnover
1st stage	6	2	2.67	10.67
2nd stage	10.67	4	4.89	19.56
Item	16.67	6.00	7.56	30.23

and
$$\frac{T}{Y} = \frac{30.23}{6.00 + 7.56} = 2.23\text{–}.$$

Thus, as the effect of the rise of prices of "basic raw materials" in relation to wages by 50 per cent, the ratio T/Y has risen from 2.0 to 2.23, i.e., only by 11.5 per cent.

Since the degree of monopoly remained constant, $(C+D+S)/Y$ has of course increased in the same proportion (from $6/(6+6)=0.5$ to $7.56/(6.00+7.56)=5.58$). It may seem at first approach paradoxical that the rise in price of foreign raw materials causes an increase of the relative share of capitalist incomes, depreciation, and salaries and thus a fall in the relative share of wages. But since the rise in prices of foreign raw materials increases the turnover T, and with a given degree of monopoly $C+D+S$ constitutes a constant percentage of T, $C+D+S$ also must increase while the wage bill remains by assumption unaltered.

4. We have seen that: (1) The rise of the degree of monopoly causes a less than proportionate increase of $(C+D+S)/Y$ (in our example 10 per cent and 7.5 per cent respectively). (2) The rise of prices of "basic raw materials" in relation to wages causes also an increase of

$(C+D+S)/Y$ but in a *much* lesser proportion (in our example 50 per cent and 11.5 per cent respectively). Thus we can find here some reasons for the tendency of relative shares towards stability. Indeed, the degree of monopoly does not undergo violent changes either in the long run or in the short period. The fluctuations in prices of "basic raw materials" in relation to wages, though strong, are only slightly reflected in the changes of relative shares. But of course if the most unfavourable case of joint action of these factors occurs, the changes of the relative shares may be appreciable; if for instance in our scheme the degree of monopoly increases by 10 per cent and the "basic raw materials" become 50 per cent dearer, $(C+D+S)/Y$ increases by about 20 per cent. We shall see below that the remarkable stability of the relative shares which we notice in statistics is the result of these determinants working in opposite directions. This phenomenon occurred only by chance during the long period considered, and may cease in the future; while in the business cycle there seems to be a steady tendency for the conflict of two forces to keep the fluctuation of relative shares within a rather narrow field.

CHANGES IN THE DISTRIBUTION OF THE NATIONAL INCOME IN THE LONG RUN

1. The degree of monopoly has undoubtedly a tendency to increase in the long run because of the progress of concentration. Many branches of industries become "oligopolistic"; and oligopolies are often transformed into cartels.

As concerns the secular trend of the ratio of turnover to income under the influence of changes in the relation of prices of "basic raw materials" to wages it is difficult to say anything definite a priori.

2. As we have seen in the first section: the relative share of manual labour in the national income in Great Britain has fallen between 1880 and 1913 from 43 to 39 and thus the relative share of capitalist incomes, depreciation, and salaries has risen from 57 to 61. The relation between Sauerbeck's index of wholesale prices and Mr. Clark's index for deflation of national income has not changed (between 1880 and 1913 both Sauerbeck's and Clark's indexes[11] increased by 6 per cent). Thus the relation T/Y has not altered and the degree of monopoly must have increased in the same proportion as the relative share of capitalist incomes, depreciation, and salaries in the national income (i.e., by 7 per cent if the figures are precise).

Between 1913 and 1935 we do not see any appreciable change in the relative shares, but the degree of monopoly increases considerably. Indeed, Sauerbeck's index has fallen during that time by 2 per cent,

[11] *National Income and Outlay*, pp. 231 ff.

while the "income prices"[12] have risen by about 60 per cent. Of course Sauerbeck's index is not suitable for representing "turnover prices," since the weight of raw materials in it is too large and that of finished goods too small, but this divergence is sufficient to show that there was a considerable change in T/Y because of the relative fall in the prices of raw materials. I tried to make a rough estimate of the rise of T/Y and I think it is unlikely to be less than 10–15 per cent. Thus the degree of monopoly has increased between 1913 and 1935 more than between 1880 and 1913. The only reason why the relative shares have not changed during the last twenty years is the sharp fall in the prices of "basic raw materials." (Between 1930 and 1935 the increase in the degree of monopoly in Great Britain seems to have been especially strong.) Should a fall in prices of basic raw materials not have happened during the period considered, the relative share of capitalist income, depreciation, and salaries in gross income would have risen during the last 20 years from 63 to 70 per cent at least, a change which amounts to a fall in the share of manual labour from 37 to 30 per cent. Had this been so, it is obvious that the economic and political face of Great Britain now would be quite different.

The development in the U. S. A. between 1909 and 1928 is similar. The relative shares were approximately stable while the value of T/Y appreciably diminished. The wholesale all-commodity index increased by about 45 per cent, King's index of "income prices" by about 80 per cent. Thus here again the degree of monopoly must have risen considerably, but the influence on the relative shares was counterbalanced by the relative fall of the prices of "basic raw materials."

It is of course not at all certain that in the future the rise of the degree of monopoly will continue to be compensated by a fall in the prices of "basic raw materials." If it fails to do so the relative share of manual labour will tend to decline.

CHANGES IN THE DISTRIBUTION OF THE NATIONAL INCOME DURING THE BUSINESS CYCLE

1. We shall examine first the changes which the ratio of turnover to income T/Y undergoes during the business cycle as a result of changes in the prices of "basic raw materials" in relation to wages.

The prices of produce of agriculture and mining fluctuate much more strongly than the cost of labour in other industries. This is due to the fact that marginal cost curves in agriculture and mining, as distinct from other sectors of the economy, slope steeply upwards. In addition, wages in agriculture fluctuate much more strongly during the business cycle than in other branches of the economy. The rise (or fall) of "basic

[12] *Ibid.*, pp. 235 and 204.

raw material" prices relative to labour cost causes, as was shown above, an increase (or decrease) in value of T/Y. Thus the value of T/Y must rise in the boom and fall in the slump.

Much more complicated is the question of the changes of degree of monopoly in the trade cycle. It was recently admitted by Mr. Harrod that the degree of monopoly increases in the boom and falls in the slump. In the slump consumers "resent and resist the curtailment of their wonted pleasures . . . Their efforts to find cheapness become strenuous and eager. Nor are commercial firms exempt from this influence upon their purchase policy; they too have received a nasty jolt and must strain every nerve to reduce costs."[13] Thus the imperfection of the market is reduced and the degree of monopoly diminished.

Mr. Harrod was rightly criticised in that there exist other factors which influence the degree of monopoly in the opposite direction. For instance, in the slump, cartels are created to save profits[14] and this of course increases the degree of monopoly, while they are afterwards dissolved in the boom because of improving prospects of independent activity and the emergence of outsiders. It must be added that the fall of prices of raw materials in the slump creates among the entrepreneurs a reluctance to "pass it on to the buyer," and this too of course increases the degree of monopoly. And it can be stated on the basis of data quoted above that the influence of these factors in raising the degree of monopoly during the slump is stronger than that of the diminishing imperfection of the market.

If we look at our data on relative shares we see that in general they do not change much during the business cycle and in any case there is no clear tendency for the relative share of manual labour to rise during the slump. But the relative share of capital income, depreciation, and salaries in the total income is equal to $(\mu)T/Y$. Thus, if the value of this expression does not fall in the slump while T/Y does as a result of the fall in the price of "basic raw materials" relative to wages, the degree of monopoly must have the tendency to *increase* in the slump and fall in the boom.

We see now that, as was already mentioned, the apparent stability of relative shares in the cycle is in reality the effect of the opposite changes of (μ) and T/Y.

2. We shall now examine the special problem of the influence of changes in money wages on the distribution of national income.

Wage cutting is likely to increase to a certain extent the degree of monopoly because a tendency may exist "not to pass it on to the

[13] *The Trade Cycle*, pp. 86–87.

[14] Joan Robinson, Review of J. F. Harrod *The Trade Cycle*, *Economic Journal*, Vol. 46, Dec., 1936, pp. 690–593.

buyer." As concerns the ratio of turnover to income, the all-round reduction of wages by the same per cent in a closed economy leaves it of course unaffected. But in an open economy this is not the case. The reduction of wages in Great Britain, e.g., importing most of the "basic raw materials," must cause a rise of T/Y. Thus both (μ) (probably) and T/Y (in an open economy importing raw materials) will be increased by the reduction of wages and consequently so will the relative share of capitalist income, depreciation, and salaries; or what amounts to the same thing, the relative share of manual labour will be reduced.

These results may be of some importance for the interpretation of the Keynesian theory of wages. This theory states *grosso modo* that the reduction of money wages in a closed system (the rate of interest being kept constant) causes a proportionate fall in prices while employment remains unaltered; for an increase in employment and income must raise the volume of saving and this must be accompanied by a rise in the volume of investment, which is however unlikely to occur. If we take into account, however, that the degree of monopoly increases as a result of wage reduction, which, as we stated above, is likely to happen, and thus the distribution of income is changed to the disadvantage of manual workers, then to the same volume of investment corresponds a lower level of employment; for the same amount can now be saved out of a smaller income. In Keynesian terms this may be expressed by saying that the fall in money wages lowers the propensity to consume by increasing the degree of monopoly and in consequence tends to reduce employment.[15]

If we pass from a closed to an open system a fall in money wages may cause an increase in the balance of trade and thus in foreign investment, and this of course raises employment. But it follows from what we have shown above that this is not the only influence appearing in an open system. For in a country importing raw materials the ratio of turnover to income will increase, and this causes a change of distribution to the disadvantage of manual workers, and consequently reduces the propensity to consume. Thus if we "open" the system there will be two opposite forces at work, and it is by no means clear in what direction they will "push" the employment.

London

[15] Mr. Keynes also considers the possibility of wage reductions influencing the propensity to consume but on other lines—e.g., by causing a shift of income from entrepreneurs to rentiers. *General Theory*, p. 262.

THE DEMAND FOR PASSENGER CARS IN THE UNITED STATES*

BY P. DE WOLFF

1. INTRODUCTION

THE PRESENT PAPER treats the results of an examination of the factors which determine the demand for motor cars in the United States. The examination is restricted to the demand for passenger cars, because for these more complete data can be obtained. Moreover the demand for trucks and buses would require a different treatment because the demand for these vehicles is exercised by an economically quite different class of buyers.

The sales of passenger cars in the home market are obtained by taking the difference between the number of cars produced and the number exported. The importation of cars into the United States can be totally neglected.

The restriction to the home market is of great importance, because in this way a more homogeneous field of investigation is obtained in consequence of which the most important factors can be determined more easily. On the other hand a substantial part of the total production is still included, e.g., 89 per cent of the 1934 production was sold in the home market.

The investigation covers the period 1921–1934. The data are chiefly taken from the publication, *Automobile Facts and Figures*, compiled by the Automobile Manufacturers' Association. Some other data are taken from official statistical publications of the United States.

In Table 1 the most important data are collected; they are graphically represented in Figure 1.

2. GENERAL LINE OF ATTACK

In Section 1 the demand for passenger cars has been defined as the difference between production and exportation. This implies the assumption that changes in the stocks of cars can be neglected. The data relating to these stocks show that the changes during the course of a whole year approximately cancel out each other and, as in this paper only annual figures will be used, the assumption seems to be justified.

By production the total number of cars every year is increased by an amount P. But on the other hand there will also be a decrease, due to

* Apart from some additions, this study is an English translation of a paper, "De vraag naar personenauto's in de Vereenigde Staten," published in *De Nederlandsche Conjunctuur*, November, 1936, p. 18.

TABLE 1

SOME FIGURES RELATIVE TO THE MOTOR CAR MARKET IN THE UNITED STATES

	Units	1921	1922	1923	1924	1925	1926	1927
Production of passenger cars for the home market	10^6 cars	1.406	2.168	3.451	2.972	3.419	3.493	2.605
Number of scrapped passenger cars	10^6 cars	0.469	0.743	0.913	1.187	1.568	1.688	1.930
Average price of passenger cars	$	719	659	606	618	657	695	735
Nonworkers' income, speculative income included	$\$10^9$	26.7	29.0	32.4	34.7	40.0	38.9	40.8
Net profit of corporations	$\$10^9$	0.46	4.77	6.31	5.37	7.62	7.50	6.51

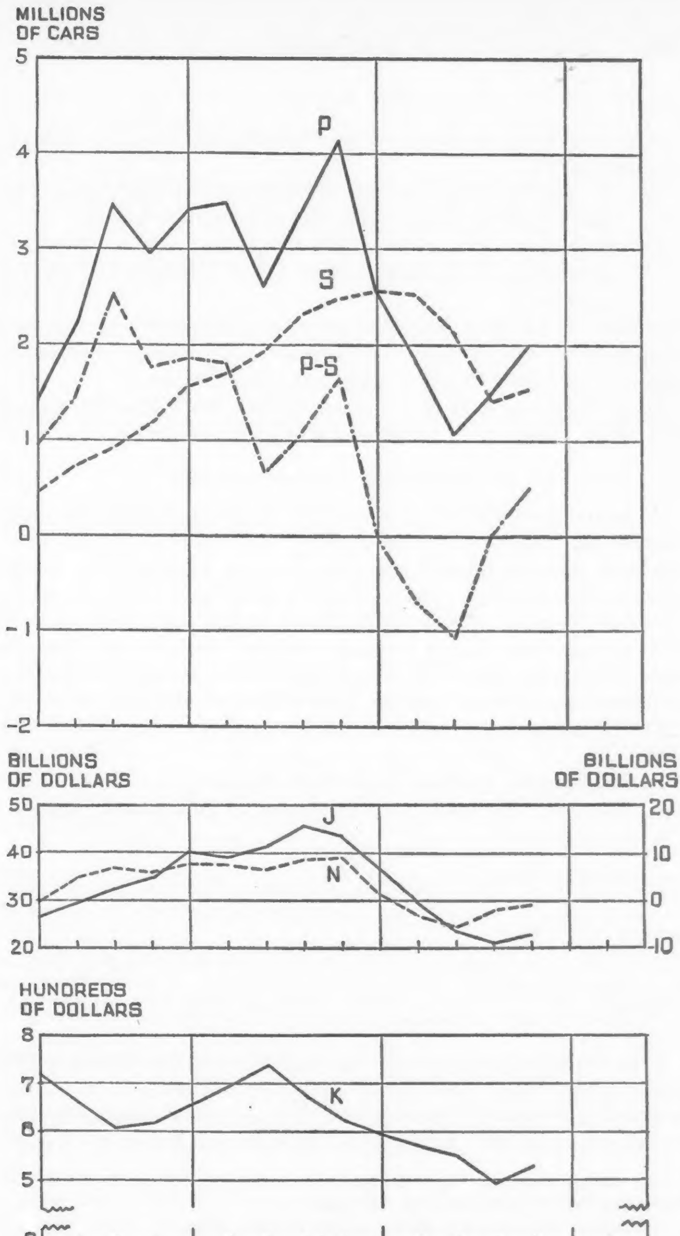
	Units	1928	1929	1930	1931	1932	1933	1934
Production of passenger cars for the home market	10^6 cars	3.395	4.141	2.541	1.839	1.063	1.475	1.994
Number of scrapped passenger cars	10^6 cars	2.317	2.501	2.568	2.538	2.144	1.416	1.544
Average price of passenger cars	$	674	622	591	566	549	489	530
Nonworkers' income, speculative income included	$\$10^9$	45.0	43.1	36.1	29.6	23.2	20.9	22.6
Net profit of corporations	$\$10^9$	8.23	8.74	1.55	−3.29	−5.65	−2.55	−1.00*

* Estimated figure.

the scrapping of old cars. In consequence of these two causes the total change in the number of cars is equal to $P-S$. This must evidently be equal to the increase in the number of registered cars ΔR. Since P and ΔR are known, S can be calculated for every year from the relation: $P-S=\Delta R$.[1] This relation can also be written in the following way:

[1] In accordance with the method for the calculation of the number of scrapped cars used in *Automobile Facts and Figures*, S is not taken equal to the value of $P-\Delta R$ for the same year but equal to the average of the values of $P-\Delta R$ for the same year and the next year. This stands in relation to the American regulations for the registration of cars and is only of minor importance for the investigation.

FIGURE 1.—P = Production of passenger cars for the home market; S = Number of scrapped passenger cars, in millions of cars; $P-S$ = Difference between P and S; J = Nonworkers' income in billion dollars, left-hand scale; N = Net profits of corporations in billion dollars, right-hand scale; K = Average price of passenger cars in hundreds of dollars.


MILLIONS
OF CARS
5
4
3
2
1
0
-1
-2
P
S
P-S
BILLIONS
OF DOLLARS
50
40
30
20
J
N
BILLIONS
OF DOLLARS
20
10
0
-10
HUNDREDS
OF DOLLARS
8
7
6
5
0
K
1921
1925
1930
1935

$$P = S + \Delta R.$$

In this form it can be interpreted as a splitting up of the total demand into two parts:

S = the number of cars needed to replace the scrapped cars. This number will be called: "The demand for replacement."

ΔR = the number of cars increasing the present number of cars in circulation. This number will be called: "The demand for first purchase."

The sign of ΔR need not necessarily be positive. During the last depression ΔR was negative, which means that the number of produced cars was not even sufficient to replace the scrapped ones.

In the following sections a separate explanation will be given for S and ΔR. P, S, and $P-S$ are shown graphically in Figure 1.

3. THE DEMAND FOR REPLACEMENT

As a car can be used only during a limited number of years one could suppose that there exists a lag between the number of cars annually produced and the number scrapped, this lag corresponding to the average lifetime of a car. But it is evident from Figure 1 that the movements of P and S are very different and in particular S shows a far more smooth form. This is probably connected with the fact that the underlying assumption of an average lifetime is too rough. The lifetime is different for different cars and there will exist a distribution of lifetimes. Investigations about these distributions have only seldom been made.[2]

In this paper the results of a recent investigation, made by Scoville,[3] have been used. His results are represented in Table 2 and Figure 2.

TABLE 2

Years	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Per cent of cars, still in use at end of year	99.1	97.9	96.1	93.2	90.2	84.4	71.0	46.9	31.5	22.2	15.3	9.7	5.2	2.3	—
Per cent of cars, scrapped during year	0.9	1.2	1.8	2.9	3.0	5.8	13.4	24.1	15.4	9.3	6.9	5.6	4.5	2.9	2.3

With the aid of this table S can be calculated. The figures in the second row of Table 2 indicate the percentages of the number of cars produced at a certain moment which will be scrapped during the first, second, etc., year after production. The value of S for 1930, e.g., can

[2] See George Gardner, "Life Tables for Automobiles," *Quarterly Journal of Economics*, Vol. 47, Feb., 1933, pp. 368–369.

[3] *Behavior of the Automobile Industry in Depression*, 1935.

now be calculated by adding the percentage of the production of 1929 for which the lifetime is one year, the percentage of the production of 1928 for which the lifetime is two years, and so on. (Attention must be drawn to the fact that these percentages are not identical with those mentioned in the second row of Table 2, because the average lifetime of the cars scrapped during the fifth year is nearly equal to $4\frac{1}{2}$ years. A good approximation of the percentage of cars having a lifetime of

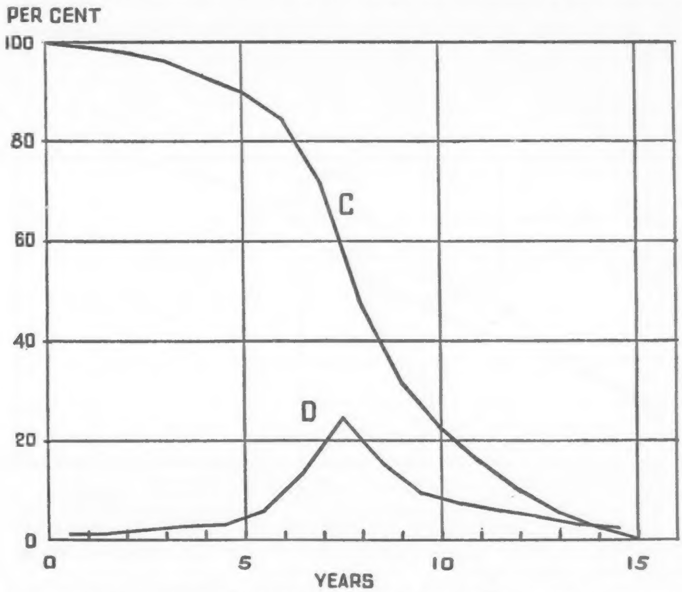


FIGURE 2.—C = Percentage of the number of cars still in use; D = Percentage of the number of cars annually scrapped.

5 years can be obtained by forming the average of the percentages scrapped during the fifth and sixth years.) In this way the number of scrapped cars, called S_b, can be calculated for every year. This quantity is shown in Figure 3. There is still a considerable difference between S_b and the true number of scrapped cars. Therefore an attempt has been made to explain the remaining differences from cyclical influences. In prosperous times the number of scrapped cars will be greater than in normal times, whereas a period of depression will have the reverse effect.

The stage of the business cycle which has been reached at a certain moment must be measured by some indicator. For this purpose dif-

ferent income data have been used, partly taken from *Statistics of Income*, partly from investigations by Warburton and others. There have been considered consecutively: total national income, nonworkers' income, speculative income, and corporation profits.[4]

The following assumptions have been made:

1. S_b will represent the number of scrapped cars, when cyclical movements are absent, i.e., when the indicator used does not deviate from its equilibrium value.

2. A deviation of the indicator from its normal value will cause a difference between S and S_b, in the same sense.

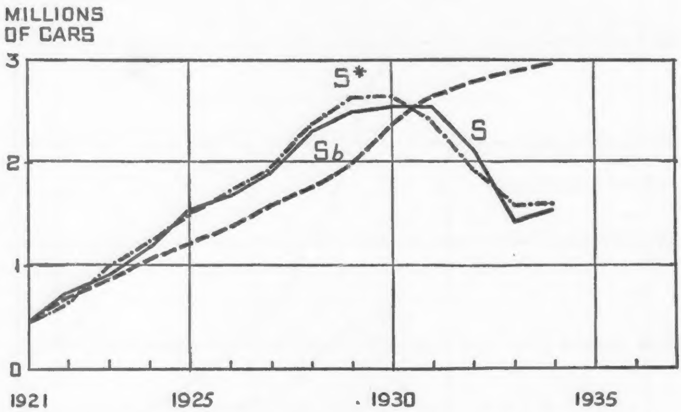


FIGURE 3.—S = True number of scrapped cars; S_b = Number of scrapped cars calculated by using the table of Scoville; S^* = Number of scrapped cars calculated by using formula (1).

3. The difference between S and S_b measured in per cent of S_b, i.e., $100(S-S_b)/S_b$, will be proportional to the deviation of the indicator from its normal value.

To check these assumptions various correlation calculations have been made between the quantity $100(S-S_b)/S_b$ and the different series of income figures. The value of $100(S-S_b)/S_b$ has not only been compared with the value of the special indicator for the same year, but also with the value for the previous year in order to introduce the possibility of time lags. Finally a linear trend has been introduced

[4] The term nonworkers' income is used here for the following combination of income figures: income from interest and dividends augmented by the estimated amount of speculative profits and half of the amount of salaries.

to take account of possible gradual changes in the "normal" value of the income.[5]

The best fit was obtained when using nonworkers' income, the correlation coefficient being 0.98. This coefficient amounted to 0.91 and 0.88 respectively in the cases where total national income and corporation profits were used as indicators. If we denote nonworkers' income earned during a certain year by J, during the previous year by J_{-1} (both expressed in thousand millions of dollars), and the number of years passed since 1921 by t, the regression equation established may be written as follows:

$$(1) \qquad 100(S - S_b)/S_b = 2.24J + 1.00J_{-1} - 98.5 - 1.64t.$$

This relation may be reduced to

$$100(S - S_b)/S_b = 3.24(0.69J + 0.31J_{-1}) - 98.5 - 1.64t.$$

Now $0.69J + 0.31J_{-1}$ is approximately equal to the nonworkers' income earned during a year, beginning ± 4 months ($=0.31$ years) before the year for which $100(S - S_b)/S_b$ has been calculated. Denoting this income by $J_{-0.31}$, the relation can be changed into

$$(2) \qquad 100(S - S_b)/S_b = 3.24[J_{-0.31} - (29.7 + 0.51t)].$$

The quantity in parentheses $(29.7 + 0.51t)$ evidently must be considered as the normal value of nonworkers' income. (Indeed it coincides practically with the expression obtained by fitting a linear trend to the nonworkers' income.)

With the help of relation (2) and of the known values of S_b and J it is possible to calculate S for the whole period. This quantity denoted by S^* is also represented in Figure 3 and it evidently agrees very well with S.[6]

4. THE DEMAND FOR FIRST PURCHASE

At first glance it is clear that the quantity $P - S$, defined in Section 2 as the demand for first purchase, shows a decreasing trend. Such gradual movements are generally caused by "long run" factors. Here a saturation process may be thought of in the first place. When an article is introduced into the market a certain time must elapse before equilibrium has been reached. In the beginning the article is demanded by only a few buyers. Gradually their number will increase up to a cer-

[5] An influence of the price of cars on the demand for replacement could hardly be discovered. A satisfactory agreement is already obtained without introducing the price at all. If the price is taken as an explaining factor instead of the nonworkers' income, the fit becomes decidedly less good.

[6] In this paper the calculated value of a quantity will be denoted by the same symbol as the actual value, but with an asterisk.

tain maximum and then again a slackening will set in. At the end of the period the market is saturated and the quantity which then still can be sold will only serve to replace worn-out units.

The period 1921–1934 is too short to distinguish whether the trend in new purchases can be interpreted as the last part of such a saturation process. Therefore the quantity $P-S$ is calculated back to 1905. The result of this calculation is shown in Figure 4.

Here the saturation phenomenon is evident in spite of the large waves which distort the shape of the curve. Curve M, the nine-year moving average of $P-S$, from which the waves mostly have been eliminated, shows this tendency still more clearly.

As it is our purpose to give an explanation of the cyclical movements of $P-S$, it will be necessary to eliminate the trend of this quantity caused by the saturation process. It would have been possible to consider the curve M itself as the trend of $P-S$. But M shows still some minor cyclical movements and according to its construction it is not known for the years 1931–1934. Therefore an attempt has been made to represent the trend by a curve with a simple mathematical expression.

Among the large class of growth curves which have been developed the logistic curve belongs to the most simple ones and as the data do not give evidence for the application of more complicated shapes the logistic has been adopted in our investigation. It is represented in Figure 5 by curve E (scale at left). (Strictly speaking curve F is the "logistic" curve and E is its derivative, but in this paper the name logistic will also be applied to E.)

The logistic curve is completely characterised by three parameters:

1. The saturation value (the distance between the axis and asymptote of curve F in Figure 5).

2. The maximum "buying" velocity (maximum value of curve $P-S$ = tangent of the angle between the tangent of curve B at its point of inflection and the axis).

3. The moment at which the maximum is reached.

The parameters have now to be chosen in such a way that the logistic curve belonging to these values fits as well as possible to curve $P-S$ in Figure 4. The parameters for which the best fit is obtained are:

Saturation value = 22.8 millions of cars;

Maximum velocity = 1.51 millions of cars a year;

The maximum value is reached approximately 1.2 months before the end of 1921.[7]

Finally we have to give an explanation of the deviations A of the new

[7] The mathematical expression can be written $T = 1.51 \cosh^{-2} 0.13t$ (t to be measured from "1920.9").

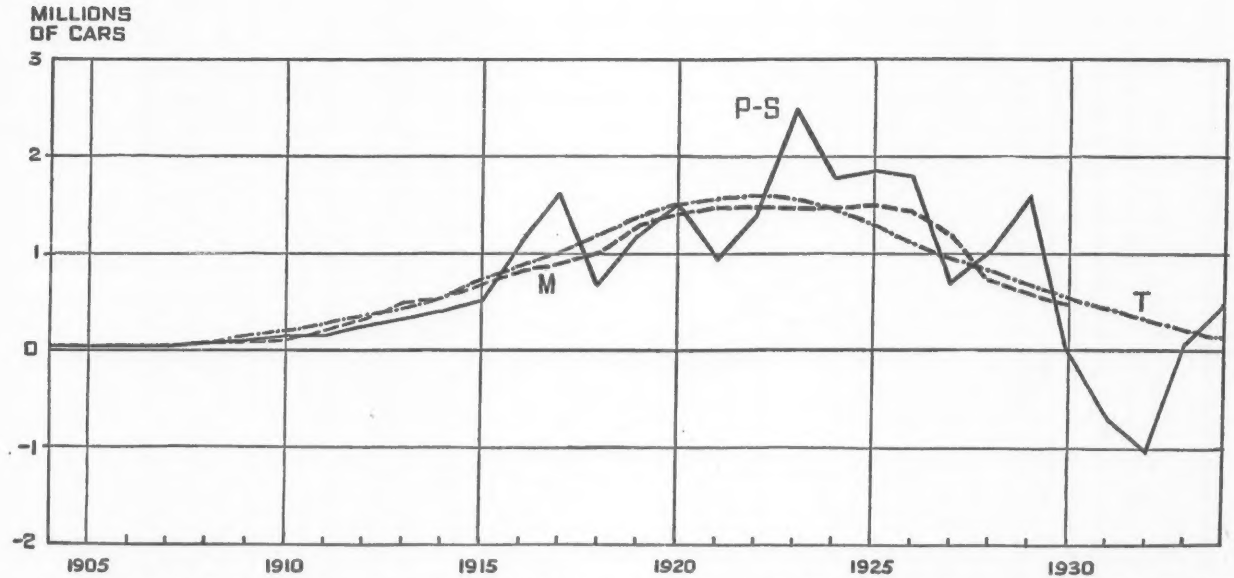


FIGURE 4.—$P-S$ = Difference between the number of produced and scrapped cars; M = Nine-years moving average of $P-S$; T = Logistic curve adapted to $P-S$.

purchases from their trend value, i.e., for the differences between $P-S$ and T in Figure 4.

The explaining factors which have been chosen in this case are the price of cars and once more one of the indicators of the phase of the cycle. The greatest difficulty is in establishing an adequate price index. In this study an average price of cars is used, calculated from the value and the number of cars produced during a year. This is a very rough method, but it is difficult to replace it by a better one.[8] More-

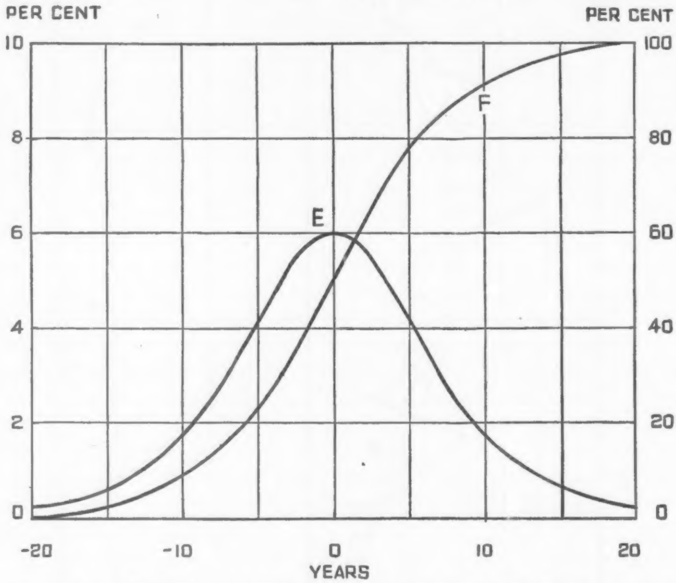


FIGURE 5.—E = Logistic growth curve, left-hand scale; F = Saturation curve corresponding to E, right-hand scale.

over it agrees satisfactorily with a price index of cars, compiled by the U. S. Bureau of Labor Statistics, starting in 1925. To represent the purchasing power the same indices have been used as in Section 3.

Correlation calculations have been carried out between the above-mentioned differences A, the price index, and one of the indicators of purchasing power. In this case the best fit was obtained when using corporation profits as an index of purchasing power. The correlation coefficient was then 0.91. Using nonworkers' income it was only slightly smaller, 0.88.

[8] Cf. Scoville, *op. cit.*, p. 13.

The regression equation obtained as a result of the first-mentioned calculation can be written as follows:

$$A = -0.65K + 0.20N + 3.36. \tag{3}$$

The symbols used have the following meaning:

A = deviations between $P-S$ and its trend, in millions of cars;
K = price of cars, in hundreds of dollars;
N = total corporation profits, billions of dollars.

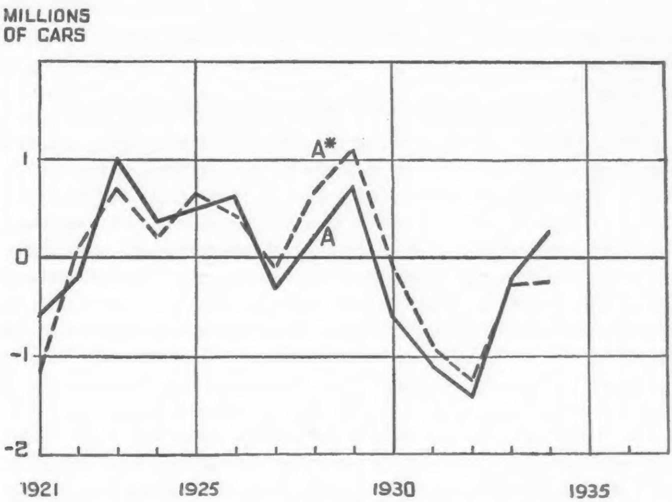


FIGURE 6.—A = True new purchases of passenger cars, deviations from trend; A^* = Calculated value of A.

The result could not be improved by introducing a time lag between N and A.

In Figure 6 the values of A, calculated from equation (3), that is A^*, are compared with the true value of A. The agreement is very good.

With the help of equation (3) it is possible to compute the elasticity of demand. In our case the quantity P has to be considered as the demand for cars, the elasticity therefore is $\epsilon = \partial P/\partial K : P/K$ (the minus sign has been left out). Now P consists of two parts, of which only $P-S$ is dependent on K and so the formula can be reduced to:

$$\epsilon = \frac{\partial A}{\partial K} : \frac{P}{K} = 0.65 \frac{K}{P} \cdot$$

In this form the formula has been used and the results are shown in Table 3.

TABLE 3

Year	1921	1922	1923	1924	1925	1926	1927	1928	1929	1930	1931	1932	1933	1934
Elasticity of total demand for passenger cars	3.3	2.0	1.1	1.3	1.3	1.3	1.8	1.3	1.0	1.5	2.0	3.4	2.2	1.7

It follows from these results that the elasticity is considerable in periods of depression; during prosperous times it decreases and in 1929 it went even down to 1 (=the limit between elastic and inelastic demand).

5. CLOSER EXAMINATION OF SOME OF THE NOTIONS USED

New purchases and replacements, in the sense in which they are used here, are related to the total number of cars present in the United States. The same concepts can be used in the case of an individual purchase. But the total number of such individual new purchases should not be confused with the number of new purchases used in this study and the same holds true for the replacements. The difference is chiefly due to the fact that a car owner can have scrapped his car without buying a new one. In our method this class of cars has been taken together with an equally great class of cars bought by persons who did not own a car before, and both classes have been considered together as replacements. So the number of replacements, in the sense used in this paper, always exceeds the number according to the other method. Although exact data concerning the difference do not exist, some idea of this difference may be got from considering the data for S and $P-S$ during the depression period. In 1932, e.g., the value of $P-S$ is equal to -1.081 million cars. But it is evident that in the other terminology the number of new purchases must be positive or at least zero and thus the number of persons who have their car scrapped during 1932 without buying a new one must at least have amounted to 1.081 millions. This proves clearly that the two concepts cannot be identified.

As to the indicators of purchasing power used the following remarks may be made: In order to obtain the best explanations of S and $P-S$, different purchasing-power indicators had to be used. In the first case nonworkers' income was used, in the second case total profits of corporations. In order to explain this divergency one might be inclined to assume an influence of business outlook on new purchases, whereas replacements should be determined chiefly by the earned income. But the difference cannot be very significant owing to the fact that corpo-

ration profits form an important part of the nonworkers' income. Moreover the explanation is nearly as good when corporation profits are used in both cases. The correlation coefficient in the explanation of S then decreases from 0.98 to 0.88 as has been explained in Section 3. Obviously a better fit is obtained with corporation profits and nonworkers' income than with total national income and workers' income. This probably will be connected with the fact that the demand for passenger cars is chiefly exercised by nonworkers.

6. CONCLUSION

The explanation of the demand for passenger cars, given in the preceding paragraphs, is a practical application of the well-known theoretical distinction between demand for replacement and demand for new purchases. The explanations given for both parts are analogous to a certain extent as they both contain a structural development disturbed by cyclical movements. Those structural developments may be considered as the trends around which the cyclical movements take place.

It is an advantage of the scheme presented here that not only have the cyclical movements been explained but also an explanation is given of the trends. Therefore it was not necessary to eliminate these trends by purely statistical methods in order to obtain the remaining cyclical fluctuations, but this could be done with the aid of technical and economic considerations. In the case of the replacements the trend was due to the restricted lifetime of cars; in the case of the new purchases the trend is caused by the saturation process of the market.

Figure 7 shows the final result of the investigation. In it are represented the true value of P and the calculated value P^*, together with its four components. (S^* does not consist of two additive parts, but is connected with S_b according to formula (2). Therefore S^* is simply split up into two parts: S_b and S^*-S_b, which are both shown in Figure 7.)

The study is related to the period 1921–1934 and although two years have passed since the end of this period the necessary data for an extrapolation of the established formulae are not yet at hand. Yet some remarks can be made now.

The quantity S_b is chiefly dependent on the production of cars 7, 8, and 9 years before and only to a very small extent on the production of cars, 1, 2, or 3 years before (see Table 2, second row). Therefore S_b can be calculated for 1935, 1936, and 1937 with a high degree of precision. The results of these calculations are given in Table 4, first row.

The value of S_b is contained between the two indicated limits and it

is seen that the uncertainty due to the fact that the values of P for 1936 and 1937 are not yet known is very small. The figures show that S_b is going to decrease in 1937. This decrease corresponds to the fall in the production of cars after 1929, but according to the distribution of lifetimes this decrease is spread out over several years and therefore the effect is very unimportant.

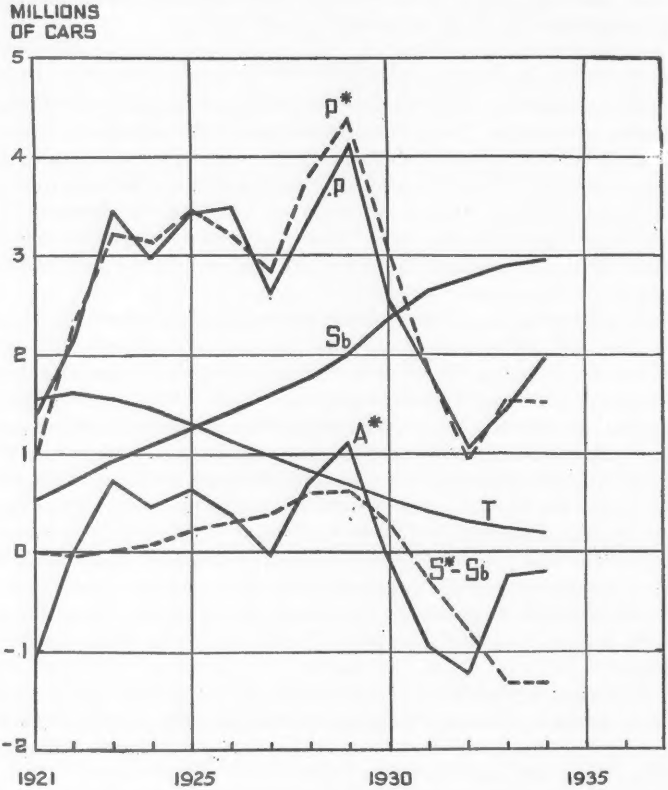


FIGURE 7.—Shares of the different factors in the explanation of P: P = True number of passenger cars produced for the home market; S_b = Number of scrapped cars, according to Scoville's lifetime distribution; $S^* - S_b$ = Surplus or deficit of scrapped cars, caused by the business cycle; T = Trend of new purchases; A^* = Deviations of the new purchases from their trend value caused by the business cycle; P^* = Calculated number of cars produced for the home market, sum of the quantities S_b, $S - S_b$, T, and A.

TABLE 4

Years	Unit	1935	1936	1937
S_b calculated from the lifetime table of Scoville	10^6 cars	3.00	3.08	2.90–2.95
S calculated from S_b, assuming hypothesis 1	10^6 cars	1.88	1.98	1.83
S calculated from S_b, assuming hypothesis 2	10^6 cars	1.88	2.33	2.63

Nonworkers' income is not yet known for 1935, but it will not be far from 27 billions of dollars. With the help of this value S is calculated on the basis of the two following hypotheses:

1. Nonworkers' income for 1936 and 1937 is assumed to be equal to the value for 1935, i.e., 27 billions of dollars. The decreasing tendency of S_b is then also reflected in S.

2. Nonworkers' income for 1936 and 1937 is assumed to have been augmented by 5 billions of dollars each year, thus 32 and 37 billions of dollars respectively.

This increase may be a little exaggerated, but it is not unrealistic. Comparable increases have occurred before, e.g., in 1924–25 and 1927–28.

In this case the decreasing tendency of S_b is overcompensated and S still shows an increase. These results are not in accordance with the so-called "echo theory" introduced by D. H. Robertson, S. de Wolff, and others. In its simplest form this theory may be formulated as follows: The means of production have a limited lifetime T. When a considerable quantity of these goods is bought at a certain moment it has to be replaced, after approximately T years (echo effect). The demand for such goods will therefore show an alternating movement with a period of T years and this in turn will cause analogous fluctuations in industrial employment. In this way the business cycle is generated. According to this theory the fall in P in 1929 had to cause a corresponding fall in S_b, 7 or 8 years later, but, as we have seen already, this fall is greatly reduced on account of the distribution of lifetimes. Moreover a small cyclical influence may be sufficient to compensate completely for the remaining decrease. At any rate the echo effect seems, at least for motor cars, not to be strong enough to cause important cyclical movements.

The results of these calculations are shown in Figure 8.

The trend value for 1935 of the new purchases can be calculated from its mathematical expression, deduced in Section 4, and amounts to 0.15 millions of cars. From this result it is seen that the market is practically saturated and in the future the "structural new purchases"

will nearly be equal to zero. But here the influence of an increasing population and of increasing wealth on the saturation value is not taken into account. A correction for the first-mentioned factor is easily made by assuming the saturation value to be proportional to population, which has increased at an average rate of 0.8 per cent per year during the last few years; a constant annual contribution to the total demand of $0.008 \times 22.5 = 0.18$ millions of cars may be expected on this account.[9]

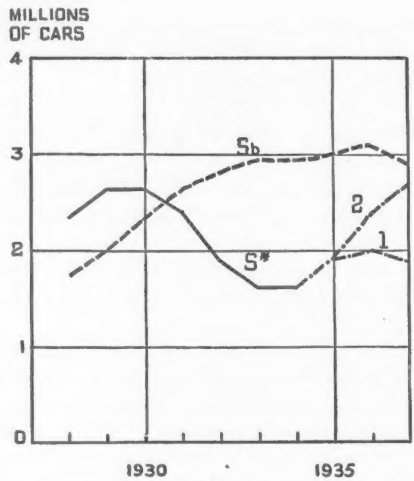


FIGURE 8.—S_b = Number of scrapped passenger cars calculated by using the table of Scoville, extrapolated for 1936 and 1937; S^* = Number of scrapped cars, calculated from S_b by using formula (2), with hypotheses 1 and 2.

The deviation A of $P-S$ from its trend T can be calculated from formula (3) if car prices K and corporation profits N are known. Now K can be calculated in the same way for 1935 as for the previous years, but N is not yet known. Even the value for 1934 had to be estimated. Such an estimate can be made by comparing the values of N with the sample index of corporation profits, published in the *Survey of Current Business* since 1928. The agreement between this index and N is satisfactory (although there may be some difference in the trends of both quantities) and in this way N can be estimated for 1934 and 1935.

[9] It must be remembered that the figure of 22.8 millions of cars is obtained from the assumption of a logistic saturation curve. Another curve would have given another, though probably only slightly different value.

The value for 1935 is equal to 4.5 billion dollars. (It is, however, not improbable that the value of -1.00 billion dollars in 1934 is too low, which would explain partly the great difference between A and A^* in that year. Cf. Figure 6.) With the help of this estimate A^* becomes:

$$A^* = -0.65 \times 5.26 + 0.20 \times 4.5 + 3.36 = 0.85 \text{ millions of cars.}$$

$(P-S)^*$, which is equal to A^*+T, becomes $0.15+0.85=1.00$ millions of cars.

From these calculations a value of P^* for 1935 can be obtained, being equal to the sum of $(P-S)^*$ and S^*;

$$P^* = (P-S)^* + S^* = 1.00 + 1.88 = 2.88 \text{ millions of cars.}$$

As the true value of P equals 3.04 millions of cars, the formulae also seem to give a correct approximation for 1935.

The formulae established in this paper are not forecasting formulae. This is brought about by the fact that the values of the explaining factors are known too late. But it is probable that these factors can be replaced by indices which can be calculated at an earlier time (this has already been done partly for N) and then it would be possible to forecast the development of the demand. Of course this would be of interest on account of the great importance of the motor-car industry to business life in the U. S. A.

In conclusion, my best thanks are due to Dr. J. Tinbergen for his encouraging interest and his valuable suggestions.

The Hague

INTERCOMMODITY RELATIONSHIPS IN STABLE DEMAND

By E. E. Lewis

Two goods may be regarded as substitutes, complements, or independent according as the compensated change in the price of one produces a similar, inverse, or negligible change in the demand for the second.[1] This, to be sure, is not the only possible criterion of intercommodity relationship in demand, but in any case it embodies an important theoretical problem. The present paper is devoted to an examination of the effect which a change in a single price is likely to have in modifying the purchases of other items in a given demand pattern.

We shall suppose that the quantities purchased are determined solely (a) by the money income of the consumer and (b) by the relative-price structure with which he is faced. Demand may be said to be *stable* as long as the functional relationship between income and relative prices on the one hand and quantities purchased on the other remains invariant.[2] It is further assumed that in each income period all income is spent upon a list of n commodities, and that the demand behavior of one consumer only is being considered. The first of these restrictions frees us from the problem of savings, and the second eliminates the complications of a changing income distribution as well as the problem of "statistical" representativeness arising from the diverse behavior of individuals.

I. BASIC RELATIONS

The dependence of quantities purchased upon money income and relative prices may be expressed by the following set of n equations:

Let u be the income of the purchaser, and p_i and x_i the price and the quantity respectively of the ith commodity. Then we may write,

$$\sum p_i x_i = u \qquad (i = 1, 2, \cdots, n) \tag{1a}$$

and $n-1$ further equations, of which the jth is

$$r_j(x_1, x_2, \cdots, x_n) = p_j/p_1 \qquad (j = 2, 3, \cdots, n). \tag{1b}$$

The first of these equations expresses the fact that all income is spent; and the remaining $n-1$ equations, the fact that if the indicated

[1] See, for example, J. R. Hicks and R. G. D. Allen, "A Reconsideration of the Theory of Value," *Economica*, Feb. and May, 1934, Part II, p. 211. Also, Henry Schultz, "Interrelations of Demand, Price and Income," *Journal of Political Economy*, Aug., 1935, p. 459.

[2] Stable demand may be regarded as a definition of demand similar in character but more comprehensive than the usual Cournot-Marshallian concept of a fixed relation between the price and quantity of a single commodity.

set of quantities is purchased, a particular set of relative prices must obtain.[3]

In exhibiting the quantity variations arising from changes in income or prices or both, it is convenient to assume that the latter changes are all functionally related to a single independent variable ω. Let $t_u = \partial u/\partial\omega$, and $t_j = \partial r_j/\partial\omega$, where as always $j = 2, 3, \cdots, n$. Further, let $s = -\sum x_i(\partial p_i/\partial\omega)$, that is, minus the change in cost of a fixed set of quantities, this change being expressed in relation to the variable ω. Differentiating equations (1a) and (1b) with respect to ω, and setting $r_{ji} = \partial r_j/\partial x_i$, we have

$$\text{(2a)} \qquad \sum p_i \frac{\partial x_i}{\partial\omega} = t_u + s$$

and $n-1$ further equations of which the jth is

$$\text{(2b)} \qquad \sum r_{ji} \frac{\partial x_i}{\partial\omega} = t_j.$$

The nth-order determinant formed by the coefficients of the partial derivatives for which these equations are to be solved is

$$R = \begin{vmatrix} p_1 & p_2 & \cdots p_m & p_{m+1} & \cdots p_n \\ r_{21} & r_{22} & \cdots r_{2m} & r_{2,m+1} & \cdots r_{2n} \\ r_{31} & r_{32} & \cdots r_{3m} & r_{3,m+1} & \cdots r_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ r_{m1} & r_{m2} & \cdots r_{mm} & r_{m,m+1} & \cdots r_{mn} \\ r_{m+1,1} & r_{m+1,2} & \cdots r_{m+1,m} & r_{m+1,m+1} & \cdots r_{m+1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ r_{n1} & r_{n2} & \cdots r_{nm} & r_{n,m+1} & \cdots r_{nn} \end{vmatrix}.$$

Let P_i be the cofactor of p_i in the first row of R and let R_{ji} be the cofactor of r_{ji}. Solving equations (2a) and (2b), we have

$$\text{(3)} \qquad \frac{\partial x_i}{\partial\omega} = (t_u + s)\frac{P_i}{R} + \frac{\sum R_{ji}t_j}{R}.$$

From this general relationship we get the following interesting cases:

i. If all prices are constant and income varies, the income derivative

[3] It is evident that the form of the equations is similar to that found in the analysis of indifference functions. (Hicks and Allen, *op. cit.*, Part II, p. 206.) We may, if we wish, regard $-r_j$ as the marginal rate of substitution of x_j and x_1, but it is sufficient for present purposes simply to regard r_j as defining a necessary price ratio.

is yielded. Noting that $s = t_j = 0$, we put $d\omega = du$ (i.e., $t_u = 1$), and write

$$\frac{\partial x_i}{\partial u} = \frac{P_i}{R}. \tag{4}$$

ii. The change in x_i arising from a change in p_1 may be expressed by putting $t_u = 0$, $s = -x_1$, $t_j = p_j/p_1^2$, and $d\omega = dp_1$. Then

$$\frac{\partial x_i}{\partial p_1} = -x_1 \frac{P_i}{R} - \frac{\sum R_{ji} p_j}{R p_1^2}. \tag{5}$$

iii. Price changes are said to be compensated if there is a simultaneous change in income equal to the change in cost of the basket of goods originally purchased. The latter is evidently $\sum x_i dp_i$—expressed relatively to $d\omega$ it is $-s$. Setting $t_u = -s$ and using primes to indicate the compensated changes in quantity, we have

$$\frac{\partial x_i'}{\partial \omega} = \frac{\sum R_{ji} t_j}{R}. \tag{6}$$

iv. In particular, the expression for quantity changes arising from a compensated change in p_1 is

$$\frac{\partial x_i'}{\partial p_1} = -\frac{\sum R_{ji} p_j}{R p_1^2}. \tag{7}$$

II. THE PARTITION THEOREMS

In discussing the interrelations of commodities in a demand pattern, it is at times necessary to select for special study subgroups of two or more items; in order to do so, a methodological device for breaking up the total demand pattern is needed. The case of compensated price changes will be discussed first.

In equation (6) the compensated change in each of the n quantities, x_i, is expressed in terms of the $n-1$ variables, t_j. By selecting the $n-2$ variables, $t_3, t_4, \cdots, t_n$, in terms of t_2 so that $\partial x_3'/\partial\omega = \cdots = \partial x_n'/\partial\omega = 0$, the quantity variations are confined to x_1 and x_2 only. Similarly, quantity variations may be confined to the first m commodities by properly expressing the $n-m$ variables, $t_{m+1}, \cdots, t_n$ in terms of the $m-1$ variables, $t_2, \cdots, t_m$. It is desired to find expressions for the quantity changes in the first m commodities (arising from compensated price changes) under the condition that the remaining $n-m$ quantities are constant.[4]

[4] Since the order of commodities may be changed by proper redesignation, it is perfectly general to treat the first m commodities; x_1, of which the price forms the denominator of r_j, must, however, remain the first commodity. But

Let the subscript g run from 2 to m, and the subscript h from $m+1$ to n; and let the subscripts G and H represent particular values in the respective ranges. Our first task is to express the $n-m$ variables, t_h, in terms of the $m-1$ "given" variables, t_g.

Setting the right-hand side of equation (6) equal to zero for $i=m+1, \cdots, n$, and putting $L_H = \sum R_{gH} t_g$ $(g=1, 2, \cdots, m)$, the condition that the last $n-m$ quantities be constant gives us $n-m$ equations, of which the Hth is[5]

$$(8) \qquad \sum R_{hH} t_h = -L_H \quad (h = m+1, \cdots, n).$$

To exhibit the solution of these $n-m$ equations in t_h, we write the adjoint of the determinant R:

$$R' = \begin{vmatrix} P_1 & P_2 & \cdots P_m & P_{m+1} & \cdots P_n \\ R_{21} & R_{22} & \cdots R_{2m} & R_{2,m+1} & \cdots R_{2n} \\ R_{31} & R_{32} & \cdots R_{3m} & R_{3,m+1} & \cdots R_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ R_{m1} & R_{m2} & \cdots R_{mm} & R_{m,m+1} & \cdots R_{mn} \\ R_{m+1,1} & R_{m+1,2} & \cdots R_{m+1,m} & R_{m+1,m+1} & \cdots R_{m+1,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ R_{n1} & R_{n2} & \cdots R_{nm} & R_{n,m+1} & \cdots R_{nn} \end{vmatrix}.$$

The $(n-m)$th-order determinant formed by deleting the first m rows and the first m columns of R' is designated by $\overline{R}'$ and the cofactor of R_{ji} in $\overline{R}'$ by ρ_{ji}.

The determinant formed by the coefficients of t_h in the $n-m$ equations represented in equation (8) is $\overline{R}'$ with the rows and columns interchanged.

If in solving equations (8) for the particular unknown, t_H, we replace the $(H-m)$th column of the determinant just mentioned by the constant terms on the right-hand side of these equations (i.e., by $-L_{m+1}, \cdots, -L_n$) and interchange the rows and columns, we get a determinant which is $\overline{R}'$ with the $(H-m)$th row replaced by these constant terms. Thus, the numerator of the solution for t_H is $-\sum \rho_{Hh} L_h$ $(h=m+1, \cdots, n)$, and,

---

this limitation is of no consequence, since the effect of a change in p_1 is being studied. The conclusions drawn from such a study are of general significance, for any commodity may be selected as x_1, that is, the relative-price structure may be determined by dividing any given price into all the others.

[5] We may assume that $R \neq 0$.

$$t_H = -\frac{\sum \rho_{Hh} L_h}{\overline{R}'} \qquad (h = m+1, \cdots, n). \tag{9}$$

By means of equation (9) each of the $n-m$ variables, t_h, is expressed as a function of the $m-1$ variables, t_g, since each L_h is a function of the latter. Thus, we may express the quantity changes in each of the first m commodities in terms of the $m-1$ variables, t_g, if we replace in equation (6) each of the $n-m$ variables, t_h, by its value as given by equation (9) and put $i=1, 2, \cdots, m$ successively. For definiteness, let us find the expression for the change in x_1.

Setting $i=1$ in equation (6) and separating the first $m-1$ terms on the right-hand side from the remaining $n-m$ terms, we may write

$$R\frac{\partial x_1'}{\partial \omega} = \sum R_{g1} t_g + \sum R_{h1} t_h. \tag{10}$$

After the substitution for each of the variables, t_h, has been made (by applying equation (9)), it is evident that the particular variable, t_G, appears $n-m+1$ times—once in the first sum on the right (with a coefficient equal to R_{G1}), and once in each of the $n-m$ substitutions for t_h. Let us designate the total coefficient of the variable, t_G, in the expression for x_1 (i.e., in equation (10)) by c_{1G}.

The coefficient of t_G in the particular expression L_H is R_{GH}. Hence by equation (9) the coefficient of t_G in the expression for the particular variable, t_H, is $-\sum \rho_{Hh} R_{Gh}/\overline{R}'$ $(h=m+1, \cdots, n)$. But in the second sum on the right-hand side of equation (10) each variable, t_H, is multiplied by R_{H1}. Thus, in getting the entire coefficient of t_G in this second sum, we multiply its coefficient in each variable, t_h, by R_{h1} and sum from $m+1$ to n. Adding the term R_{G1} (from the first sum on the right-hand side of equation (10)) and using h and h' to distinguish the two summations, we may write

$$c_{1G} = \frac{\overline{R}' R_{G1} - \sum_{h'=m+1}^{n} \sum_{h=m+1}^{n} R_{h'1} \rho_{h'h} R_{Gh}}{\overline{R}'}. \tag{11}$$

To identify the numerator of this expression, we write the following $(n-m+1)$th order determinant, which is found by adding to $\overline{R}'$ the appropriate elements of the Gth row and the first column of R':

$$d_{1G} = \begin{vmatrix} R_{G1} & \cdots & R_{G,m+1} & \cdots & R_{Gn} \\ R_{m+1,1} & \cdots & R_{m+1,m+1} & \cdot & R_{m+1,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ R_{n1} & \cdots & R_{n,m+1} & \cdots & R_{nn} \end{vmatrix}.$$

Developing this by the first column, we find that the minor of the first element is $\overline{R}'$, and that the minor of each of the remaining elements consists of a determinant which is $\overline{R}'$ with the corresponding row replaced by the elements (except the first) of the first row of d_{1G}. Thus, the minor of $R_{m+1,1}$ may be written $\sum R_{Gh}\rho_{m+1,h}$ $(h=m+1, \cdots, n)$, and similarly for the remaining elements in the first column. In each case the sign of the cofactor is minus. For the minor of the pth element in the column is put in the indicated form by transposing $p-2$ times the elements of the first row of d_{1G}. Multiplying the elements of the first column by their cofactors and adding, we have

$$d_{1G} = \overline{R}'R_{G1} - \sum_{h'=m+1}^{n} \sum_{h=m+1}^{n} R_{h'1}\rho_{h'h}R_{Gh}. \tag{12}$$

Comparing equations (11) and (12), we have

$$c_{1G} = \frac{d_{1G}}{\overline{R}'}. \tag{13}$$

The denominator of this expression is found by deleting the first m rows and the first m columns of R'. Let $\overline{R}$ be the determinant formed by deleting the last $n-m$ rows and the last $n-m$ columns of R. Then the well-known theorem on adjoint determinants[6] shows that

$$\overline{R}' = \overline{R}R^{n-m-1}. \tag{14}$$

Since d_{1G} is found by deleting from R' the first m rows except the Gth and the first m columns except the first, the same theorem shows that

$$d_{1G} = \overline{R}_{G1}R^{n-m}, \tag{15}$$

where $\overline{R}_{G1}$ is the cofactor of r_{G1} in $\overline{R}$. Hence, from equation (13),

$$c_{1G} = \frac{\overline{R}_{G1}R}{\overline{R}}. \tag{16}$$

Since the same procedure may be applied to the coefficients of the rest of the variables, t_g, in equation (10), the expression for the change in x_1 when the last $n-m$ quantities are held constant is given by

$$\frac{\partial x_1'}{\partial \omega} = \frac{\sum \overline{R}_{g1}t_g}{\overline{R}}. \tag{17}$$

In a word, the expression has the same form as equation (6). We simply delete from R the rows and columns which refer to prices and quantities of the commodities held constant.

[6] See, for example, Bôcher, *Introduction to Higher Algebra*, p. 31

A similar expression for the change in any other of the first m quantities may be found in precisely the same fashion.

An analogous theorem may be proved for the response of the purchaser to a change in resources, this response being confined to the first m commodities. Suppose that the relative prices of this latter group do not change (i.e., $t_g=0$). Suppose, further, that the remaining $n-m$ price ratios change in such a way that the *uncompensated* effect of these price changes upon the $n-m$ quantities, x_h, is zero. Under these conditions, the amount of money "saved" upon the last $n-m$ commodities, because of the changes in their cost, will be "transferred" to or from the first m commodities. The quantity changes in the latter furnish the desired information.

Let us express the change in x_1 in relation to the amount of money $(-\sum x_h dp_h)$ thus transferred, designated by $d\alpha$. Setting $t_u=t_g=0$, $d\omega=d\alpha$, and hence $s=1$ in equation (3), we have

$$\frac{\partial x_i}{\partial \alpha} = \frac{P_i}{R} + \frac{\sum R_{hi}t_h}{R} \qquad (h = m+1, \cdots, n). \tag{18}$$

The condition that the last $n-m$ quantities be constant yields $n-m$ equations, of which the Hth is

$$\sum R_{hH}t_h = -P_H \qquad (h = m+1, \cdots, n). \tag{19}$$

As before, the determinant formed by the coefficients of t_h in these equations is equal to $\overline{R}'$. Likewise, if in solving for t_H we replace the proper column by the terms on the right-hand side of equation (19), and interchange the rows and columns, we get a determinant which is $\overline{R}'$ with the $(H-m)$th rows replaced by $-P_{m+1}, \cdots, -P_n$. Thus,

$$t_H = -\frac{\sum \rho_{Hh}P_h}{\overline{R}'} \qquad (h = m+1, \cdots, n). \tag{20}$$

Setting $i=1$ in equation (18), the expression for the change in x_1 may be written

$$R\frac{\partial x_1}{\partial \alpha} = \overline{R}'P_1 - \frac{\sum_{h'=m+1}^{n}\sum_{h=m+1}^{n} R_{h'1}\rho_{h'h}P_h}{\overline{R}'}. \tag{21}$$

The numerator of this expression may be shown to be the $(n-m+1)$th-order determinant formed by adding to $\overline{R}'$ the appropriate elements in the first row and the first column of R'. Converting to the minors of R by the theorem on adjoint determinants, and setting $\overline{P}_1$ as the cofactor of p_1 in $\overline{R}$, we have

$$\frac{\partial x_1}{\partial \alpha} = \frac{\overline{P}_1}{\overline{R}}. \tag{22}$$

In the same way we may find a similar expression for the quantity change in any other of the first m commodities.

Thus, the expression for the response of the purchaser to changes in resources with respect to the first m commodities is of the same form as the income derivative of equation (4).

III. THE TWO-COMMODITY PARAMETERS

If two commodities, x_1 and x_j, alone are allowed to vary in quantity, the effect of a compensated change in p_1 reflects what may be termed the binary competitive relationship of the two. Setting $d\omega = dp_1$ and $g = 1, j$ in equation (17), we have[7]

$$\frac{\partial x_1'}{\partial p_1} = \frac{\dfrac{p_j^2}{p_1^2}}{\begin{vmatrix} p_1 & p_j \\ r_{j1} & r_{jj} \end{vmatrix}}. \tag{23}$$

It is convenient to designate second-order determinants like that in the denominator above by the symbol $r_a(bc)$, the letters in parenthesis indicating the prices of the first row and the corresponding partials of the given price ratio.

Using v_1 and v_j to designate the respective changes in x_1 and x_j, each relative to the change in p_1, we have,

$$v_1 = \frac{\partial x_1}{\partial p_1} = \frac{p_j^2}{p_1^2 r_j(1j)} \tag{24}$$

and

$$v_j = \frac{\partial x_j}{\partial p_1} = \frac{-p_j}{p_1 r_j(1j)}. \tag{25}$$

The relationship of two commodities may also be characterized by the changes in their quantities arising from a shift in spending to or from the given pair, when their price ratio remains constant and other quantities are unchanged. In this way, the nature of the "binary association" of the pair is exhibited.

Let us assume that the amount of money so shifted, $d\alpha$, is related to the change in p_1 by the equation $d\alpha = b\,dp_1$.[8] Then from equation (22) it follows that

$$v_1' = \frac{\partial x_1}{\partial p_1} = \frac{br_{jj}}{r_j(1j)} \tag{26}$$

[7] If $j \neq 2$, shift the jth row and the jth column of R each $j-2$ times before applying equation (17). This double shift leaves the sign of R unchanged.

[8] Since the price ratio of x_1 and x_j is constant, the equation $d\alpha = b\,dp_1$ implies that p_j changes proportionally with p_1. The reason for this particular formulation of the problem will appear later.

and

$$v_j' = \frac{\partial x_j}{\partial p_1} = \frac{br_{j1}}{r_j(1j)} \cdot \tag{27}$$

If there is no competitive relation at all between two commodities (one may question whether such a situation ever exists), then from equations (24) and (25) it is evident that $r_j(1j)$ is infinite. But in such a case either r_{j1} or r_{jj} or both must be infinite, and v_1' and v_j' will in general have finite values, one of which may be zero.

The degree of binary competition and the nature of the binary association of two commodities completely characterize their relationship when they alone are allowed to vary. But if the $n-2$ remaining quantities change along with those of x_1 and x_j, an additional kind of influence must be taken into account. For the relationship between the relative consumption of x_1 and x_j and their price ratio may be modified by changes in other quantities. It is necessary, therefore, to calculate the shift in spending between x_1 and x_j produced by changes in other quantities, under the condition that the price ratio of these is unchanged (i.e., $dr_j=0$).

Quantity changes again being related to dp_1, the conditions laid down may be expressed as follows:

$$p_1 \frac{\partial x_1}{\partial p_1} + p_j \frac{\partial x_j}{\partial p_1} = 0, \tag{28}$$

$$r_{j1} \frac{\partial x_1}{\partial p_1} + r_{jj} \frac{\partial x_j}{\partial p_1} = - \sum r_{jk} \frac{\partial x_k}{\partial p_1} = b', \tag{29}$$

where the subscript k refers to the $n-2$ commodities other than x_1 and x_j.

Hence

$$v_1'' = \frac{\partial x_1}{\partial p_1} = \frac{-p_j b'}{r_j(1j)} \tag{30}$$

and

$$v_j'' = \frac{\partial x_j}{\partial p_1} = \frac{p_1 b'}{r_j(1j)} \cdot \tag{31}$$

IV. THE COMPONENTS OF $\partial x_1'/\partial p_1$ AND $\partial x_j'/\partial p_1$

With the foregoing descriptive measures of the relationship of x_1 and x_j, we may now turn to the main problem of our discussion, the effect of a compensated change in p_1 upon the demand for x_j. For definiteness, suppose p_1 declines, the change being compensated.

There being no restrictions upon quantity variation in the present instance, the change in p_1 will in general affect the purchase not only

of x_1 and x_j, but also of the $n-2$ remaining items in the budget pattern. Taking the latter $n-2$ quantity changes for granted, we shall examine the increments dx_1' and dx_j'. Under the conditions laid down, these increments may each be regarded as made up of three components:

i. A shift in spending between x_1 and x_j arising from the change in their price ratio. This reflects the binary competitive relationship of the pair, and will in general be from x_j to the (cheapened) commodity x_1.

ii. A shift in spending to or from the pair, x_1 and x_j, resulting from the fact that the quantities but not the prices of the $n-2$ other commodities vary.

iii. A shift in spending between x_1 and x_j resulting from the fact that changes in other quantities may modify the relation of the relative consumption of the pair to its price ratio.

Expressions for these components will be presented, each relative to dp_1, and then it will be shown that the three components for each of the two commodities add up respectively to $\partial x_1'/\partial p_1$ and $\partial x_j'/\partial p_1$.

The first pair of components, reflecting the binary competitive relationship of x_1 and x_j, are evidently v_1 and v_j defined in equations (24) and (25) respectively.

In deriving the second pair of components, suppose that the price ratio of x_1 and x_j is unchanged, and that precisely the same amount of money is shifted to (or from) x_1 and x_j, taken together, as is true in the case being discussed. This amount of money is $-\sum p_k dx_k'$, the subscript k again indicating the $n-2$ commodities other than x_1 and x_j.

In expressing the ratio of this amount of money to the change in p_1, it is convenient to take $w_i = \sum R_{ji} p_j$. Then, from equation (7),

$$\frac{\partial x_i'}{\partial p_1} = \frac{-w_i}{Rp_1^2}. \tag{32}$$

Thus,

$$-\sum p_k \frac{\partial x_k}{\partial p_1} = \frac{\sum p_k w_k}{Rp_1^2} = \frac{d\alpha}{dp_1} = b. \tag{33}$$

Substitution in equations (26) and (27) yields the second pair of components:

$$v_1' = \frac{r_{jj} \sum p_k w_k}{r_j(1j) Rp_1^2} \tag{34}$$

and

$$v_j' = \frac{-r_{j1} \sum p_k w_k}{r_j(1j) Rp_1^2}. \tag{35}$$

The third pair of components is yielded by substituting from equation (32) in equations (30) and (31):

(36) $$v_1'' = \frac{-p_j \sum r_{jk} w_k}{r_j(1j)Rp_1^2}$$

and

(37) $$v_j'' = \frac{\sum r_{jk} w_k}{r_j(1j)Rp_1}.$$

In showing that $v_1+v_1'+v_1''=\partial x_1'/\partial p_1$, we multiply each of the indicated terms by $r_j(1j)Rp_1^2$, and write the resulting equality, which is to be justified. Thus,

(38) $$p_j^2R + r_{jj}\sum p_k w_k - p_j \sum r_{jk} w_k = -r_j(1j)w_1.$$

The second and third terms on the left-hand side may be combined into $\sum r_j(kj)w_k$ and transposed. Since $r_j(1j) = -r_j(j1)$,

(39) $$p_j^2R = \sum r_j(jk)w_k + r_j(j1)w_1.$$

To justify this equality, we write the following determinant:

$$d' = \begin{vmatrix} 0 & r_j(j1) & r_j(j2) & \cdots & r_j(j, j-1) & 0 & r_j(j, j+1) & \cdots & r_j(jn) \\ 0 & p_1 & p_2 & \cdots & p_{j-1} & p_j & p_{j+1} & \cdots & p_n \\ p_2 & r_{21} & r_{22} & \cdots & r_{2,j-1} & r_{2j} & r_{2,j+1} & \cdots & r_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_j & r_{j1} & r_{j2} & \cdots & r_{j,j-1} & r_{jj} & r_{j,j+1} & \cdots & r_{jn} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_n & r_{n1} & r_{n2} & \cdots & r_{n,j-1} & r_{nj} & r_{n,j+1} & \cdots & r_{nn} \end{vmatrix}$$

The cofactor of $r_j(j1)$ is $-w_1$. For if the minor of $r_j(j1)$ is developed by its first column, it equals $\sum R_{j1}p_j = w_1$. For any other nonzero element in the first row of d', say $r_j(jp)$, the minor may be converted into w_p by transposing the first column of the minor $p-1$ times, and hence the sign of each cofactor is minus. Thus, $-d'$ is equal to the right-hand side of the equality.

If the second row of d' is multiplied by r_{jj} and the $(j+1)$st row by $-p_j$, and the results added to the first row, the first element in this row becomes $-p_j^2$ and the remaining elements zero. Since the cofactor of $-p_j^2$ is R, $-d'$ is also equal to the left-hand side of the equality. The latter is therefore justified.

In similar fashion, the components of $\partial x_j'/\partial p_1$ yield the equality

(40) $$p_1p_jR = \sum r_j(1k)w_k + r_j(1j)w_j.$$

This equality may be justified by writing a determinant similar to d' except for the first row. In the first row of this new determinant, the first two elements are zero. The element above p_2 is $-r_j(12)$, and simi-

larly for the rest of the first row. As it stands, the determinant may be shown (as above) to equal the right-hand side of the equality. It can be converted into the left-hand side by (a) multiplying the second row by $-r_{j1}$ and the $(j+1)$st row by p_1 and (b) adding the results to the first row.

V. CONCLUSIONS

In terms of the components into which $\partial x_j'/\partial p_1$ has been broken, it is possible to make certain comments concerning this "cross-commodity" relationship and in particular to examine the factors determining its sign.

The component v_j reflects the direct competitive relationship of the two commodities, and we may take its sign as positive (a cheapening, say, of x_1 producing, if anything, a decrease in purchase of x_j). Concerning it two points may be made. In the first place, when we classify two goods such as beef and lamb as "substitutes" (i.e., as responsive to changes in their relative costs), it is with our attention concentrated upon these two to the exclusion of other items in the consumer's budget. The "partition" method of confining quantity variations to the two commodities in question simply makes more precise the ordinary method of passing judgment. In the second place, the component appears in the general expression, whatever the character of the commodities involved. While the component may equal zero (the two commodities being noncompetitive), one would expect at least some response to changes in relative cost for almost any pair of goods. Thus, for example, even such closely associated commodities as bread and butter will no doubt be consumed in somewhat different proportions if the price of butter rises or falls sharply as compared with the price of bread.

The component, v_j', reflecting the change in spending upon the pair x_1 and x_j, depends upon the relation of these two to other members of the demand pattern; and hence it is difficult to anticipate the character of this term. One observation, however, seems justified.

If x_j is less competitive in the binary sense with x_1 than any of the remaining commodities, the cheapening of x_1 will call for a smaller shift in the relative consumption of these two than in that of x_1 and any other item. In such a case, the chances are good that spending will be shifted *to* x_1 and x_j, and v_j' will be negative (the cheapening of x_1 increasing the purchase of x_j). The component , v_j, though positive will be small; and if v_j'' is neglected for the moment, the total change in x_j resulting from a compensated cheapening of x_1 will be a positive increment. The two would be deemed "complements" by the simple "price-demand" test of relationship mentioned at the beginning of this paper. In fact, it would appear that when we classify goods like bread

and butter as "complements" (anticipating the price-demand relationship) it is on the basis of a lack of competitive relationship between them. Conversely, if x_j is not only competitive with x_1 but *more* competitive than other items, v_j will be large and positive; and the chances are good that less spending will be devoted to the pair, v_j' being positive as well. In this case, still ignoring v_j'', the cheapening of x_1 will decrease the demand for x_j, and the two will be counted as "substitutes" by the "price-demand" test.

In striking cases, it may be possible to anticipate the total effect of a compensated change in p_1 upon x_j on the basis of the general characteristics of the commodities involved. We are probably safe, for example, in expecting a compensated drop in the price of chicken to reduce the demand for pork, because the latter is very likely more competitive with chicken than any other commodity, and because changes in other items are not likely to obscure this relationship (i.e., v_j'' is in fact negligible). But for the general run of commodities this is obviously not the case—the question, if important, is one for empirical investigation alone.

Concerning the character of v_j'' very little can be said in advance. But if our general conclusion is that the sign of $\partial x_j'/\partial p_1$ is of doubtful significance, the existence of this uncertain term adds weight to the finding.

In any case, the sign of this derivative reflects the relative strength of certain *quantitative* influences, not whether the relationship of x_1 and x_j is of one *kind* or another. From a theoretical standpoint, this is the most serious objection to the use of the sign in classifying goods as substitutes *or* complements *or* independent.

Howard University

ON THE SIGNIFICANCE OF PROFESSOR DOUGLAS' PRODUCTION FUNCTION*

By Horst Mendershausen

In his most interesting book *The Theory of Wages*[1] Professor Douglas indicated coefficients for a production function of the American manufacturing industries lumped together. He assumed that the function was of the form:

$$(1) \qquad P = bL^kC^{1-k},$$

where P represents the volume of production, L, the volume of labor and C, the volume of fixed capital, b and k constants.

On the basis of time series for the variates and proceeding by a "modification of the method of least squares" Professor C. W. Cobb calculated the coefficients b and k contained in this equation and obtained:

$$(2) \qquad P' = 1.01L^{.75}C^{.25},$$

where P' is the computed or "theoretical" value of P.

The coefficients in this equation were interpreted as empirically found for this set of three variates. Professor Douglas used these coefficients for determining the marginal elasticity of labor and capital. He drew various conclusions from their numerical values, among others for the effect of changes in wage rates on employment.[2]

The purpose of this paper is to show that equation (2) does not give a reliable description of a real production function. In the first place the theoretical setup has been based on more than doubtful assumptions, one of them leading to an arbitrary method of verification, and in the second place the data fail partly to represent what they ought to. Finally, I think that the series used offer no access at all to the determination of a production function.

In order to make the different kinds of objections quite clear, let us consider the material used and the assumptions made in Professor Douglas' analysis.

For P, the *volume of production* from 1899 until 1922, E. E. Day's

* I am very grateful for suggestions and assistance which I found at the Oslo University Economic Institute during the preparation of this paper. I wish to express my sincere thanks to Professor R. Frisch and to his assistants, Mr. T. Haavelmo and Mr. O. Reiersöl.

[1] P. H. Douglas, *The Theory of Wages*, New York, 1934. The part of the book with which we are mainly concerned in the following has already been presented in a provisional form as a paper of C. W. Cobb and P. H. Douglas, "A Theory of Production," *American Economic Review*, Supplement, March, 1928.

[2] Douglas, *op. cit.*, p. 501.

index has been used.[3] Let us take this series as a good approximation to the facts it shall represent.

Before considering the following two series representing the volumes of labor and capital respectively, we propose to distinguish the volumes of both factors, which year for year have been *used* in production (L_u and C_u) from the volumes which were *present* (at disposal) during these years (L_p and C_p). Professor Douglas intended to represent L_u and C_u by his series: L and C.

For L, the *volume of labor* used in production, Professor Douglas computed an index by combining year for year an index for the number of clerical and wage workers employed with his index of average weekly labor hours. Since variations of employment and of the length of the standard working week have been taken into account, one may consider the L series as a good approximation to L_u, though the short-time variations of the volume of labor really used in production may have been somewhat greater than the L series indicates, in consequence of not taking overtime and part-time work into account. This remark was made by Professor Douglas himself.[4]

The figures which Professor Douglas used for the *volume of fixed capital* (C) depend to a high degree on pure estimation.[5] Professor Douglas interpolated the yearly value of machinery, implements, and equipment between two Census statistics (1899 and 1904) and one Census estimate (1922). The value of factory buildings, added to this, was obtained by interpolation between 1899 and 1904 and by extrapolation afterwards. Interpolations and extrapolations were arranged according to changing ratios of increase which equally were determined by estimates. The series for the value of fixed capital was finally deflated by a cost index.

Apart from the highly estimative character of this series the fact must be stressed that no allowance has been made for variations of the degree to which the fixed capital present was *utilised* at the different points of time. The series cannot be supposed to represent the fixed capital used (C_u); it tends to represent the fixed capital present (C_p) during the successive years. Professor Douglas fully recognized this fact; he said:

> The index does not, of course, measure the short-time fluctuations in the amount of capital used. Thus, no allowance is made for the capital which is allowed to be idle during periods of business depression or for the greater than normal intensity

[3] *Ibid.*, p. 128.

[4] *Ibid.*, pp. 128 and 135.

[5] Cf. S. H. Slichter's contribution in the discussion of the Cobb-Douglas paper quoted in note 1, *American Economic Review*, Supplement, March, 1928, pp. 166 and 167.

of use in the form of second shifts, etc., which characterizes the periods of prosperity.[6]

Since, nevertheless, Professor Douglas intended to compute coefficients for the function:

$$(3) \qquad P = f(L_u, C_u)$$

and not for a function:

$$(4) \qquad P = f(L_u, C_p),$$

he had to assume, "that each factor would have a constant degree of intensity of use from year to year."[7] Thus, he put for each year:

$$(5) \qquad C_u = C_p,$$

which, however, he himself recognized as being "to some degree false." Because this is false, the computations were actually done for function (4), where actual production is expressed as a function of—a rather good approximation of—labor *used*, and of fixed capital *used and idle*. We shall see that this is of decisive importance for the interpretation of the coefficients in the light of the data used.

Since a *constant* production function was assumed to have ruled during the 24-year period from 1899 to 1922[8] and its coefficients were to be computed from the data, it had also to be assumed that no technical progress and no change in the physical productive efficiency of an average worker had occurred during this period. Otherwise, variations in the relative volumes of both factors might not be due to changes of their combination under a constant production function but to successive transitions from one production function to another; in other words: *permanent* instead of *provisional* changes in the combination of the factors would be observed. These assumptions are manifestly in contradiction to all that economists know about the industrial development during this period. Professor Douglas made strong reservations with respect to the realism of his hypothesis of a constant production function.[9] But nevertheless he maintained it without advancing decisive arguments in its favor. The reiterated references to the good fit of the function[10] to the data is in fact a *petitio principii*, since only if this hypothesis is justified can the function claim to be taken as a *production* function.

But let us assume for a moment that Professor Douglas was right in assuming a constant production function for the American manu-

[6] *Ibid.*, p. 122.
[7] *Ibid.*, p. 132.
[8] *Ibid.*, p. 145.
[9] *Ibid.*, p. 211.
[10] *Ibid.*, for instance, pp. 150 and 210.

facturing industries during the 24 years; is it then possible to take (2) as a trustworthy expression for this function? That is the question we will now try to answer.

Professor Douglas further assumed that the production function had to correspond to a *pari passu*[11] law: "If we increase both labor and capital by a factor m, we increase the product by the same proportion."[12] Thus, the scale of production was assumed to have no influence on the relative volumes of labor and capital used. It seems that this law was not derived from the material analysed. The relation of the coefficients of labor and capital(that is to say, of the exponents in (1)) was fixed a priori in a way which permitted this law to be maintained, whether it corresponded to reality or not: their sum had to be equal to 1. Thus the coefficients of (2) were not obtained from a set of three variates: P, L, and C (that is from their respective logarithms), but from a set of *two* variates. Several possibilities existed of choosing these two variates. For instance, one could determine statistically the coefficient of L, say k, and make the coefficient of C, say k', equal to $1-k$, or determine statistically k' and put $k=1-k'$. In the first case the two variates considered would be P/C and L/C, in the second case, P/L and C/L. In Professor Douglas' analysis the first alternative was chosen without discussion.[13] The coefficient k and the constant b were statistically determined for:

$$(\log P - \log C) = b + k(\log L - \log C). \tag{6}$$

He did not fit the equation:

$$\log P = b + k \log L + k' \log C \tag{7}$$

(where k and k' are free coefficients) to the data. In this way the coefficients of (2) were derived. The result from this regression analysis in two variates was presented as a verification of the *three*-dimensional correlation.

This procedure stands or falls with the presence or absence of the assumed *pari passu* law in the reality reflected by the material. If it can be proved that the data really justify this assumption, the coefficient k will not differ if derived from (6) or from (7); in other words

[11] The *passus coefficient* is:

$$\epsilon = \frac{d \log P}{d \log L} = \frac{d \log P}{d \log C},$$

the differentials being taken for a *proportional* change in L and C. Cf. R. Frisch, *Inledning til Produksjonsteorien.* (Mimeographed lectures at Oslo University, 1937, p. 77 ff.)

[12] Douglas, *op. cit.*, p. 132.

[13] See below, note 14.

k' derived from (7) will then equal $1-k$; then and only then may one be allowed to take the coefficients from (6) over to (7). If this cannot be proved, it is erroneous or at least arbitrary to determine the coefficients of (7) in this way; erroneous, if an *ultra passum* law is found to have ruled the production function, arbitrary, if one cannot ascertain what kind of law existed in reality.

Professor Douglas did not try to furnish this proof. Nevertheless the elements for doing so are present in the material if we face it in the same way Professor Douglas did. It is only necessary to compute k' in (7) as a partial regression coefficient and to see whether it is equal to $1-k$, k being the regression coefficient in (6). The present writer has made the necessary computations for (7) and obtained the elementary regression coefficients,

$$k = 0.76 \quad \text{and} \quad k' = 0.25, \tag{8}$$

by minimizing in the log P direction. Since these coefficients are—practically speaking—identical with those found by following Professor Douglas' procedure[14] (namely $k=0.75$; $1-k=0.25$), one might think that his hypothetical *pari passu* law was in plain harmony with the material.

That is, however, too rash a conclusion. If one looks closer into the three-set (1. log P, 2. log L, 3. log C), one finds that it is a nearly perfectly *multicollinear set.*[15] The whole situation is very well exhibited by the bunch map in Figure 1.

Each pair of the three variates shows high correlation:

$$r_{12} = 0.91,\ r_{13} = 0.93,\ r_{23} = 0.91; \tag{9}$$

that is to say, the three two-sets come near to being (simply) collinear. In the three-set, therefore, more than one linear relation exists. By minimizing—in the three-set—in the three different directions one gets very different intercoefficients, and k and k' show correspondingly different values if taken from the three different elementary regression equations:

	Minimization direction	Coefficients k	k'
(10)	log P	0.76	0.25
	log L	−1.06	−1.14
	log C	2.23	−0.34

[14] Since this is the case, one would not find different exponents for L and C, if the two-set P/L, C/L had been chosen for the regression analysis.

[15] The general problem of multicollinearity has been admirably treated in R. Frisch's two publications: "Correlation and Scatter in Statistical Variables," *Nordisk Statistisk Tidskrift*, Vol. 8, 1929 (particularly §§6 and 7); *Statistical Confluence Analysis by Means of Complete Regression Systems*, Oslo, 1934.

Each of the three equations "fits " the data very well, in the sense of a high coefficient of multiple correlation: $R_{1.23}=0.94$, $R_{2.13}=0.94$, $R_{3.12}=0.93$; but none of the three pairs of regression coefficients can give an answer to the question: How does log P vary as a linear function *both* of log L and log C? They cannot, because each of them depends mainly on the *disturbances* and not on the systematic connection in the set.[16] There does not exist *one single* determinate systematic relation in this set of *three* variates.

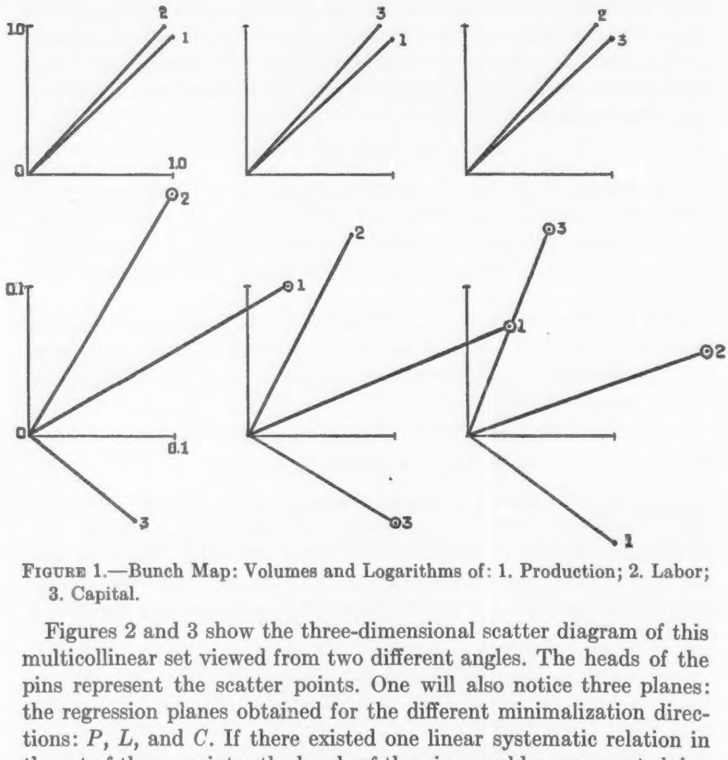


FIGURE 1.—Bunch Map: Volumes and Logarithms of: 1. Production; 2. Labor; 3. Capital.

Figures 2 and 3 show the three-dimensional scatter diagram of this multicollinear set viewed from two different angles. The heads of the pins represent the scatter points. One will also notice three planes: the regression planes obtained for the different minimalization directions: P, L, and C. If there existed one linear systematic relation in the set of three variates the heads of the pins would come near to lying in a plane. When and only when the heads were highly scattered along

[16] Professor Cobb in his paper "Production in Massachusetts Manufacturing, 1890–1928," *Journal of Political Economy*, Dec., 1930, pp. 705–707, computed diagonal, instead of elementary, regression coefficients. But the present difficulty cannot be overcome in this way. The diagonal coefficients are as misleading as the elementary ones if the set is "exploded."

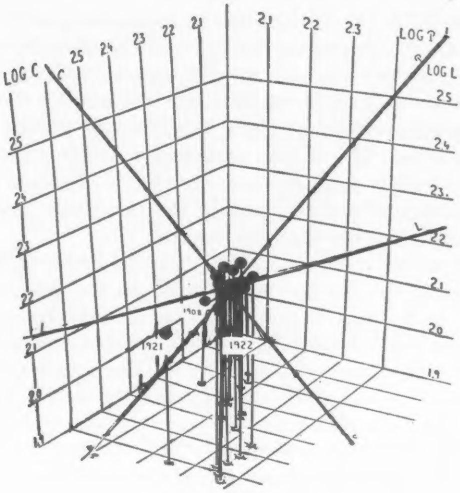


FIGURE 2

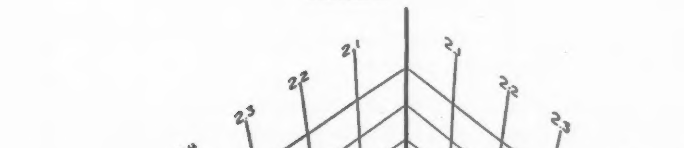


FIGURE 3

this plane, would it have a meaning to determine partial regression coefficients from this set; for only then would a statistically determined regression plane (and consequently its intersection lines with the three co-ordinate planes) be sufficiently warranted by the data. Instead of being scattered along a plane, it will be seen that the pin heads cluster about a *line*. This is seen particularly well in Figure 2 where all the heads virtually appear as one single head. There are only a few points and particularly three, namely the pin heads marked 1908, 1921, and 1922, which deviate from this line.

If one nevertheless endeavours to fit planes to the heads, these three exceptional values play a predominant role in the determination of the planes.[17] For the different minimalization directions one gets highly different planes which appear like windmill wings in Figs. 2 and 3.

The importance of the role played by the three main deviations in the determination of the coefficients can easily be shown. We take the terms 1908, 1921, and 1922 out of the three variates and determine a bunch map (Figure 4) and coefficients (r^*, k^*) for the 21 observations.

Each pair of variates now shows still higher correlation:

(15) $$r_{12}^* = 0.94, \; r_{13}^* = 0.95, \; r_{23}^* = 0.99;$$

the bunch map of the three-set shows still stronger "explosion" and the regression coefficients become:

(16) $k^* = 0.96$ (instead of $k = 0.76$ in the case of 24 observations),
$k'^* = 0.13$ (instead of $k' = 0.25$ in the case of 24 observations).

Thus the omission of 12 per cent of the terms increases the coefficient k by 26 per cent and decreases k' by 48 per cent. The whole system of

[17] The equations determining the three regression planes are:

(11) (P plane) $F_1 \equiv \log P - 0.76 \log L - 0.25 \log C + 0.01 = 0$,

(12) (L plane) $F_2 \equiv 0.45 \log P - \log L + 0.16 \log C + 0.81 = 0$,

(13) (C plane) $F_3 \equiv 0.87 \log P + 0.93 \log L - \log C - 1.60 = 0$.

The straight line which fits the scatter points so well can approximately be determined by the intersection of any pair of these planes. (We say "approximately" because the multicollinearity is not perfect.) But this line cannot *only* be found by means of a pair of these three planes. Any pair of planes can be used, under the condition that each plane contains this line. The family of planes obeying this condition is approximately determined by the general equation:

(14) $$F_i + \lambda F_j = 0, \qquad i, j = 1, 2, 3, \; i \neq j,$$

where λ may have any value. This shows that in the case of perfect multicollinearity an *infinite* number of regression planes would give a perfect "fit" to the scatter points—under any minimalization direction. Since in our case, however, disturbances prevent the two-sets from becoming perfectly collinear, there appear three definite planes which give the best fit *to the disturbances* for the respective minimalization direction. We have "fictitious determinateness," in Professor Frisch's terminology.

regression planes again reveals its instability, if considered in this way.

What are the *facts* behind the three large deviations which influence to such a high extent the coefficients k and k'? They correspond to two years of depression (1908 and 1921) and to one year of revival where apparently important technical progress occurred (1922). During the

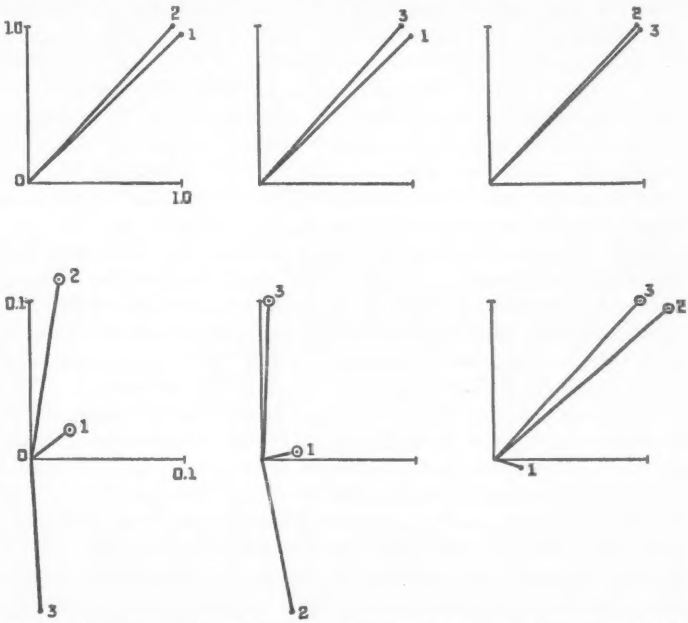


FIGURE 4.—Bunch Map: Volumes and Logarithms of: 1. Production; 2. Labor; 3. Capital.

depression years production and labor used were reduced; the volume of fixed capital *used* in production was probably reduced as well, but the volume of fixed capital *present* could not suddenly be reduced. At any rate, it was not according to Professor Douglas' C-series.[18] Consequently the fact that capital appears exceptionally high as compared with labor and production is probably due to a particularly great inequality of C_p and C_u during these years; in other words *to a particularly high degree of nonfulfilment* of one of Professor Douglas' main assumptions. It has nothing to do with variations within a constant production function. In 1922 the volume of labor used recovered less than production, while the volume of fixed capital present continued

[18] Douglas, *op. cit.*, diagram on p. 128.

its upward trend. As far as the reasons mentioned do not equally apply to this year, technical progress appears to have changed the production function; this is excluded by Professor Douglas' hypotheses.

It is now obvious that the empirical relation found between the coefficients k and k' cannot be taken as a verification of the *pari passu* law. The material does not allow to say whether Professor Douglas' assumption of a *pari passu* law was justified or not. The nature of the production law cannot be ascertained at all from this set of variates. Therefore it must be considered as arbitrary to put k and $1-k$ as determined for (6) equal to k and k', respectively, in (7). By putting $k'=2-k$, $2.5-k$, or any other number minus k, one could "verify" all imaginable "production laws" from this material—and always obtain a good fit to the data. Each of those "empirically found" coefficients would be just as unwarranted by the facts as the others are.

In other words: the value of k computed for (6)—0.75—may correctly be used as an empirically found coefficient for the two-set: P/C and L/C. But the value of k (and $1-k$) cannot be regarded as trustworthy for the three-set P, L, C, because the absence of any determinate three-dimensional regression in this set[19]—apart from that produced by disturbances—prevents an empirical justification of the assumed *pari passu* law. Only such a justification—or a separate determination of the *passus* coefficient by other data—would allow conclusions from the coefficients of the two-set as to those of the three-set.

Beside the production function for the United States manufacturing industries (2) Professor Douglas gave similar functions for Massachusetts (1890–1926 and 1890–1920) and New South Wales (1901–1927).[20] The exponents were computed in the same way. A superficial examination of the data used for these computations suggests that our criticism of the United States formula very probably applies also to these cases.

[19] Since one main reason for the multicollinearity in the three-set is the correlation of each of the variates with "time," one could be induced to try to turn this difficulty by correlating trend ratios of P, L, and C instead of their absolute values. (Cf. Henry Schultz's criticism and suggestions in his paper "Marginal Productivity and the General Pricing Process," *Journal of Political Economy*, Oct., 1929, pp. 540–541.) Professor Douglas has also made such an attempt (*op. cit.*, pp. 143–145). But he did not make use of the results of this procedure, and this appears to have been right. The deficiencies of the individual series, in particular of the C series, will in the course of this procedure become still more sensible. While the C series may be taken as a certain approximation to the trend of fixed capital, the position of its yearly values with respect to their trend cannot be relied upon. In a letter to the author, Professor Douglas says: "The statistics for New South Wales and Massachusetts are based on annual censuses in which capital figures are given for each year, which makes the problems of interpolation and of deflation much less difficult."

[20] Douglas, *op. cit.*, pp. 161 and 169.

One may finally ask whether the coefficients k obtained for the two-sets in these different cases represent any real phenomenon. Yes, they do. Each of these k's may be interpreted as a ratio between two trend-slope differences, giving a certain aspect of the *industrial progress* during the period covered (e.g., the increasing productive power of the engaged workers, or the increase in the capital intensity of the industry). In this ratio the difference between the trend slopes of production and capital is compared with the difference in the trend slopes of labor and capital.

Let us take for example the United States case. Each of the three variates ($\log P, \log L, \log C$) can be represented with rather high exactitude[21] as a linear function of time.[22]

$$\log P = \alpha t \qquad (\alpha = 0.0156), \tag{17}$$

$$\log L = \beta t \qquad (\beta = 0.0112), \tag{18}$$

$$\log C = \gamma t \qquad (\gamma = 0.0281). \tag{19}$$

The differences of the trend slopes of P and C, and L and C are:

$$(\log P - \log C) = \alpha - \gamma, \tag{20}$$

$$(\log L - \log C) = \beta - \gamma. \tag{21}$$

The ratio

$$\frac{\alpha - \gamma}{\beta - \gamma} \tag{22}$$

has to be equal to k in (6) if one is right in assuming that k just gives a ratio of the two trend-slope differences. The ratio (22) indeed turns out to be 0.741, while $k = 0.75$ (or with 3 decimals: 0.748). Thus, practically,

$$k = \frac{\alpha - \gamma}{\beta - \gamma}. \tag{23}$$

The constant k indicates whether labor and fixed capital increased during the 24 years at the same average rate or not. In our case they did not; fixed capital increased more than labor. The industry *progressed* and became more "capital-intensive."

I think that the values of k obtained by Professor Douglas for the different cases as well as the differences between these k's can only be looked upon in this way, namely as an expression for the trend in the technical development.

Oslo

[21] $r_{Pt} = 0.91$; $r_{Lt} = 0.89$; $r_{Ct} = 1$.

[22] All variates are taken here as deviations from their means.

THE MAXIMIZATION OF UTILITY OVER TIME[1]

BY GERHARD TINTNER

A. THE DISCONTINUOUS CASE: UTILITY AS A FUNCTION

SUPPOSE provisionally the existence of a utility function F. Let us assume further that the individual in question consumes only three goods x, y, and z. The argument can easily be extended later to any number of commodities. The individual is at the point in time 0 and plans for the period of time $1, 2, \cdots, n$. Consumption takes place at these n discontinuous points in time. We denote by x_j the quantity of the commodity x which the individual *plans* to consume at the point in time j; similarly, y_j and z_j. He does not accumulate any commodity stocks. The utility function F will depend on those quantities which the individual expects to consume:

$$F = F(x_1, \cdots, x_n, y_1, \cdots, y_n, z_1, \cdots, z_n). \tag{1}$$

Let us further denote by p_j the price of x *expected* for the point in time j; similarly, q_j and r_j. Let I_j be the money income *expected* for the point in time j, and E_j the total *expected* expenditure in j; then

$$E_j = x_jp_j + y_jq_j + z_jr_j. \tag{2}$$

Finally let s_j be the saving *planned* for the point in time j. Let us further assume that $s_n = 0$, i.e., that there is no saving expected in the last interval and the total income is consumed in the total period 1 to n. Let i_j denote the rate of interest *expected* for the point in time j.

We then have the following n equations between the I, incomes, E, expenditures, and s, savings which follow from their definitions:

$$\begin{aligned} I_1 &= E_1 + s_1, \\ I_2 + s_1(1 + i_1) &= E_2 + s_2, \\ &\cdots\cdots\cdots \\ I_n + s_{n-1}(1 + i_{n-1}) &= E_n. \end{aligned} \tag{3}$$

The problem is to maximize the function F under the subsidiary conditions (3).

[1] Paper given at the Third Annual Research Conference of the Cowles Commission for Research in Economics, Colorado Springs, July, 1937. The author has to thank Mr. J. R. Hicks, of Gonville and Caius College, Cambridge, for many stimulating discussions on this subject. The ideas and methods used are very similar to those used by Hicks in his article on the rate of interest, "Wages and Interest," *Economic Journal*, 1935, pp. 456 ff., and also in a forthcoming book on theoretical economics by the same author.

This is the same as maximizing a function G, defined as:

$$(4)\quad G = F + \lambda_1[I_1 - E_1 - s_1] + \lambda_2[I_2 + s_1(1 + i_1) - E_2 - s_2] + \cdots + \lambda_n[I_n + s_{n-1}(1 + i_{n-1}) - E_n]$$

where the λ's are Lagrange multipliers. The unknowns are the $3n$ quantities x, y, and z and the $(n-1)$ quantities s. There are also the n multipliers to be determined with the help of the system (3) which contains n equations.

Differentiating G with respect to the x, y, z, and s and remembering the relation (2) we get the following $4n-1$ equations:[2]

$$(5)\quad \begin{aligned} \frac{\partial G}{\partial x_j} &= \frac{\partial F}{\partial x_j} - \lambda_j p_j, && (j = 1, 2, \cdots, n) \\ &\cdots\cdots\cdots\cdots \\ \frac{\partial G}{\partial s_j} &= -\lambda_j + \lambda_{j+1}(1 + i_j), && (j = 1, 2, \cdots, n - 1) \end{aligned}$$

and similarly for y and z. Using the last $n-1$ equations of the system (5) we get for any λ, i.e., the expected marginal utility of money at the point in time j:

$$(6)\quad \lambda_j = \frac{\lambda_{j-1}}{(1 + i_{j-1})} = \frac{\lambda_1}{(1 + i_1)(1 + i_2)\cdots(1 + i_{j-1})}.$$

The expected marginal utility of money at the point in time j equals the marginal utility of money at point 1 discounted by the expected interest rates in the interval $1 \cdots j$.

And hence we can write according to (5) for every j:

$$(7)\quad \frac{\partial F}{\partial x}\frac{1}{p_j} = \frac{\partial F}{\partial y}\frac{1}{q_j} = \frac{\partial F}{\partial z}\frac{1}{r_j} = \frac{\lambda_1}{(1 + i_1)(1 + i_2)\cdots(1 + i_{j-1})}.$$

In words: The "weighted" marginal utilities of the quantities of the commodities x, y, and z which the individual expects to consume at the point in time j must be equal to the discounted marginal utility of money as described above. (3) can also be written:

[2] The fact that we get the same number of equations as we have unknowns does not, in itself, establish a sufficient, but only a necessary condition for the solution. See A. Wald, in *Ergebnisse eines mathematischen Kolloquiums*, edited by K. Menger, Vienna, 1935, 1936.

$$
\begin{aligned}
&I_1 + \frac{I_2}{(1+i_1)} + \frac{I_3}{(1+i_1)(1+i_2)} + \cdots \\
&\quad + \frac{I_n}{(1+i_1)(1+i_2)\cdots(1+i_{n-1})} \\
(8) \qquad &= E_1 + \frac{E_2}{(1+i_1)} + \frac{E_3}{(1+i_1)(1+i_2)} + \cdots \\
&\quad + \frac{E_n}{(1+i_1)(1+i_2)\cdots(1+i_{n-1})} \cdot \text{ (Budget Equation)}
\end{aligned}
$$

Treating (8) as the subsidiary condition instead of (3) we reach the same result for the relative maximum of the function F as described in (7). It is easily seen that the solution is identical with equation (7).

B. THE CONTINUOUS CASE: UTILITY AS A FUNCTIONAL

The transition from the discontinuous to the continuous case is easy, if we conceive the number of intervals between the points in time 0 and n to increase indefinitely.[3] The function F becomes a functional:

$$
(9) \qquad f = f\left[\begin{matrix} n & n & n \\ x(t), & y(t), & z(t) \\ 0 & 0 & 0 \end{matrix}\right].
$$

This corresponds to equation (1) in our former treatment. Instead of (8) we get

$$
(10) \qquad \int_0^n I(t) \exp\left[-\int_0^t \rho(s)ds\right]dt = \int_0^n E(t) \exp\left[-\int_0^t \rho(s)ds\right]dt;
$$

where $\rho(t)$ is the force of interest.

The expected expenditure at the point t, $E(t)$, becomes

$$
(11) \qquad E(t) = x(t)p(t) + y(t)q(t) + z(t)r(t).
$$

Let $f_x'[x, y, z; u]$ be the functional derivative of f at the point u, and similarly $f_y'[x, y, z; u]$ and $f_z'[x, y, z; u]$. The variation (or differential) of f is

$$
(12) \quad \delta f = \int_0^n (f_x'[x, y, z; u]\delta x + f_y'[x, y, z; u]\delta y + f_z'[x, y, z; u]\delta z)du
$$

[3] See the concept of the income stream in I. Fisher, *Theory of Interest*, New York, 1930.

δx, δy, δz being the variations in x, y, and z at the point u. In equilibrium δf must be identically equal to zero for all forms of δx, δy, δz subject to the condition:

$$\int_0^n \{p(u)\delta x(u) + q(u)\delta y(u) + r(u)\delta z(u)\} \exp\left[-\int_0^u \rho(s)ds\right]du = 0. \tag{13}$$

But this is the same as finding the maximum of a functional g:

$$g = f[x(t), y(t), z(t)] - \Lambda \int_0^n \{p(t)x(t) + q(t)y(t) + r(t)z(t)\} \exp\left[-\int_0^t \rho(s)ds\right]dt, \tag{14}$$

where Λ is again a Lagrange multiplier. Λ may be taken as a constant.[4] This gives the following equilibrium conditions analogous to (7):

$$\frac{f_x'[x, y, z; t]}{p(t)} = \frac{f_y'[x, y, z; t]}{q(t)} = \frac{f_z'[x, y, z; t]}{r(t)} = \Lambda \exp\left[-\int_0^t \rho(s)ds\right], \tag{15}$$

where again Λ is the marginal utility of money at the point in time 0 and can be determined from the conditional equation (10). The equations (15) must hold true for every point in time t between 0 and n.

Our ultimate results do not involve the marginal utilities or functional marginal utilities themselves, but only their ratios. Thus an index,

$$u = \psi\left\{f\left[\begin{matrix} n & n & n \\ x(t), & y(t), & z(t) \\ 0 & 0 & 0 \end{matrix}\right]\right\}, \tag{16}$$

will yield the same results as f, under the same condition.

C. THE MARGINAL RATES OF SUBSTITUTION

The marginal rates of substitution are then defined as:

$$R(x_j, x_k) = \frac{\partial F}{\partial x_j} : \frac{\partial F}{\partial x_k}, \tag{17}$$

and similarly for x and y, y and z also.

[4] This problem has been discussed by the late Professor Hans Hahn of Vienna in *Sitzungsberichte Akad. Wien*, Vol. 131. See also Professor C. F. Roos, "A Dynamical Theory of Economic Equilibrium," *Proc. Nat. Ac. Sci.*, Vol. 13; "A Mathematical Theory of Depreciation," *Am. Jour. Math.*, Vol. 50.

In terms of marginal rates of substitution the equilibrium condition (7) for the discontinuous case appears as follows:

$$(18)\qquad \frac{R(x_j, x_1)}{p_j} = \frac{R(y_j, x_1)}{q_j} = \frac{R(z_j, x_1)}{r_j} = \frac{m_1}{(1+i_1)(1+i_2)\cdots(1+i_{j-1})}$$

where m_1 is the marginal rate of substitution of x_1 for money at the point in time 1. The functional marginal rates of substitution R are themselves functionals of x, y, and z:

$$(19)\qquad R[x(t)] = f_x'[x, y, z; t]:f_x'[x, y, z; 0],$$

and similarly for y and z.

The equilibrium conditions now become:

$$(20)\qquad \frac{R[x(t)]}{p(t)} = \frac{R[y(t)]}{q(t)} = \frac{R[z(t)]}{r(t)} = M \exp\left[-\int_0^t \rho(s)ds\right].$$

In this formula M stands for the marginal rate of substitution of money at the point in time 0.

D. CONCLUDING REMARKS

It would be possible to extend the treatment of the discontinuous and the continuous case further along the lines of Hicks-Allen elasticity of substitution. If their definition of complementarity and independence of goods were adopted the reasoning would in the continuous case involve the limiting values of some determinants consisting of the second-order derivatives of the utility functional similar to certain determinants which occur in integral equations. The same is true for the second-order conditions, which have been worked out in the simpler case of utility as a function by Hotelling.[5] These conditions would secure a true maximum instead of a minimum or minimax of the utility functional and they involve certain inequalities on the determinants mentioned above.

Another question is the statistical verification of the theory developed above. The statistical results obtained in the simple case of utility as a function, in the earlier works of Frisch[6] and especially the extensive study of family budgets by Allen and Bowley,[7] are rather encouraging.

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[5] *Econometrica*, Vol. 3, 1935, pp. 71 ff.
[6] *New Methods of Measuring Marginal Utility*, Tübingen, 1932.
[7] *Family Expenditure*, London, 1935.

THE INFLUENCE OF DISTRIBUTED LAGS ON KALECKI'S THEORY OF THE TRADE CYCLE

By R. W. James and M. H. Belz

Some considerable interest has been created by the publication of Kalecki's[1] simple mathematical theory of the trade cycle, and it is considered in some quarters that such a commendably simple model might well be made the basis for more elaborate studies. For this reason it appears to be of interest to examine what happens when the technological structure postulated approximates a little more closely to reality than is the case with Kalecki's original model.

In Kalecki's system all capital goods have the same "gestation period," this period being considered independent of the particular phase of the cycle in which production is taking place. Tinbergen[2] has examined what modifications may be expected when the gestation period varies with the particular condition of trade, being a linear function of the volume of goods in process, but in Tinbergen's scheme, as in Kalecki's, the gestation period at any one time is the same over all industry. It is the purpose of this short note to examine how the system must be modified to take account of the fact that different goods have different periods of gestation.

We shall suppose that of all capital goods, $I(t-\theta)$, ordered[3] per unit of time at time $t-\theta$, a proportion $l_\theta \Delta\theta$ have periods of gestation lying between θ and $\theta+\Delta\theta$. The total volume of goods completed per unit of time at time t is thus $\sum l_\theta I(t-\theta)\Delta\theta$, where the summation extends over all possible values of θ. In the limit, as $\Delta\theta \to 0$, this expression takes on the limiting form $\int_0^\infty l_\theta I(t-\theta)d\theta$. Thus, in place of Kalecki's relation,

$$K'(t) = I(t-\theta) - U,$$

which expresses the fact that the rate of increase of total capital equipment, $K'(t)$, is equal to the rate of output of capital goods less the rate of replacement, U (considered constant), we have

$$K'(t) = \int_0^\infty l_\theta I(t-\theta)d\theta - U. \tag{1}$$

The volume of goods in process at time t having periods of gestation between θ and $\theta+\Delta\theta$ is given by

$$A_\theta(t)\Delta\theta = \frac{l_\theta \Delta\theta}{\theta}\int_{t-\theta}^{t} I(t')dt',$$

[1] Kalecki, Econometrica, Vol. 3, p. 327

[2] Tinbergen, Econometrica, Vol. 3, p. 268.

[3] In Kalecki's model there is no production of producers' goods in anticipation of demand. All production is in response to orders.

so that the total volume of goods of all periods in process is

$$A(t) = \int_0^\infty A_\theta(t)d\theta = \int_0^\infty \frac{l_\theta d\theta}{\theta} \int_{t-\theta}^t I(t')dt'. \tag{2}$$

Differentiating $A(t)$ with respect to time, we have

$$A'(t) = \int_0^\infty \frac{l_\theta d\theta}{\theta} [I(t) - I(t-\theta)]. \tag{3}$$

Substituting (1) and (3) in Kalecki's fundamental relation

$$I'(t) = mA'(t) - nK'(t), \qquad (m \text{ and } n \text{ constant}), \tag{4}$$

we find, as the integro-differential equation characterizing the system,

$$\begin{aligned} I'(t) &= mI(t) \int_0^\infty \frac{l_\theta}{\theta} d\theta - m \int_0^\infty \frac{l_\theta I(t-\theta)}{\theta} d\theta \\ &\quad - n \int_0^\infty l_\theta I(t-\theta) d\theta + nU. \end{aligned} \tag{5}$$

It is not necessary to carry out an exhaustive investigation of this equation to show that there is a long-wave solution corresponding to that found in the simpler system, at any rate, with ordinary types of gestation-period distribution curves. Let us assume that (5) is satisfied by the substitution

$$I(t) - U = ke^{\rho t} + \bar{k}e^{\bar{\rho}t},$$

where $\bar{k}$ and $\bar{\rho}$ are the complex conjugates of k and ρ. It is evident from simple substitution that such a relation is only possible when ρ satisfies the equation

$$\rho = m \int_0^\infty \frac{l_\theta}{\theta}(1 - e^{-\rho\theta})d\theta - n \int_0^\infty l_\theta e^{-\rho\theta} d\theta. \tag{6}$$

By way of illustration let us assume that gestation periods are distributed according to the simple Pearson curve,

$$l_\theta = \frac{\theta}{{\theta_0}^2} e^{-\theta/\theta_0} \tag{7}$$

where θ_0 is the mode of the distribution. Substituting this value of l_θ in (6), and performing the required integration, ρ is found to be given by the cubic equation

$$\rho\theta_0(1 + \rho\theta_0)^2 = m(1 + \rho\theta_0)^2 - (1 + \rho\theta_0)m - n\theta_0. \tag{8}$$

If we assume that $m = 0.95$, $n = 0.121$, $2\theta_0 = 0.6$ years,[4] the values as-

[4] The mean value of the distribution is twice the modal value.

signed to these parameters by Kalecki, the three roots of (8) are, -3.45, $-0.024 \pm 0.625\sqrt{-1}$. Hence a possible solution of (5) is

$$I(t) - U = Ce^{0.024t} \cos (0.625t + \phi),$$

where C and ϕ are constants, and the unit of time taken is the year.

The period of this oscillation is very close to 10 years, the period obtained in the simple theory, the only effect of the lag's being distributed being to introduce a slight damping into the solution. It would appear then, that so far as the long-wave solution is concerned the "spread" of gestation periods has only a minor influence.

The problem cannot be treated with such analytical simplicity for other types of gestation-period distributions, but it may be seen from a simple argument that a long-wave solution of approximately the same period is to be anticipated for all types of distribution.

To see this, develop $I(t-\theta)$ as a Taylor series about the point t, and substitute for $I(t-\theta)$ in (5) the series so obtained. This gives

$$\begin{aligned} I'(t) &= m\int_0^\infty \frac{l_\theta}{\theta}\left[\theta I'(t) - \frac{\theta^2}{2} I''(t) + \cdots \right]d\theta + nU \\ &\quad - n\int_0^\infty l_\theta\left[I(t) - \theta I'(t) + \frac{\theta^2}{2} I''(t) - \cdots \right]d\theta. \end{aligned} \tag{9}$$

This equation may be written in the form

$$\begin{aligned} &nJ(t) + J'(t)\left(1 - m - n\int_0^\infty \theta l_\theta d\theta\right) \\ &\quad + \tfrac{1}{2}J''(t)\left(m\int_0^\infty \theta l_\theta d\theta + n\int_0^\infty \theta^2 l_\theta d\theta\right) + \cdots = 0 \end{aligned} \tag{10}$$

where $J(t) = I(t) - U$.

If all derivatives of higher order than the second may be neglected, (10) reduces to an equation of the form

$$aJ''(t) + bJ'(t) + cJ(t) = 0, \qquad (a, b, \text{ and } c \text{ constants}), \tag{11}$$

a solution of which is

$$J(t) = Ce^{-bt/2a} \cos\left(\frac{\sqrt{4ac - b^2}}{2a} t + \phi\right); \tag{12}$$

if $4ac > b^2$. the values of c, b, and a are found by equating them to the first three coefficients of the development (10).

In order to test this approximation method of arriving at a long-wave solution, let us compare the result it yields for the distribution (7) with the precise result just obtained.

For all distributions we must have the relation $\int_0^\infty l_\theta d\theta = 1$, for this is equivalent to the statement that the proportion of goods with gestation periods in the range $(0, \infty)$ is unity—obviously all periods must lie in that interval. In addition, $\int_0^\infty \theta l_\theta d\theta = 2\theta_0$, for this parameter is the mean value of the distribution. By direct calculation it may be verified that $\int_0^\infty \theta^2 l_\theta d\theta = 6\theta_0^2$. We thus find, $a = \theta_0(m + 3n\theta_0) = 0.318$, $b = 1 - m - 2n\theta_0 = -0.0226$, $c = n = 0.121$. Substituting these values in (12) we find the periodic solution

$$J(t) = Ce^{-0.036t} \cos (0.616t + \phi),$$

giving a period of about 10 years as before.

Consider now the distribution

$$l_\theta = \frac{\pi}{4\theta_0} \sin \frac{\pi}{2} \cdot \frac{\theta}{\theta_0}, \quad \text{when } 0 \leqq \theta \leqq 2\theta_0, \quad \text{and}$$

$$l_\theta = 0 \text{ for all other values of } \theta.$$

The mean and the mode are identical,[5] and there are no productive processes occupying a longer period than twice this mean value. For this distribution b and c have the same values as before, but as $\int_0^\infty \theta^2 l_\theta d\theta$ now equals $1.19\theta_0^2$, $a = 0.622/2$, which parameters give a slightly damped, 10-year period, as before.

It would thus appear that the form of the frequency distribution of gestation periods has only a slight influence on the period and damping of the long-wave solution. This justifies the assumption at the basis of Kalecki's work that it is permissible to replace a frequency distribution by some sort of mean value, at least to a first approximation.

ERRATUM

Attention is drawn to a misprint in our paper, "On a Mixed Difference and Differential Equation," published in ECONOMETRICA, Vol. 4, Apr., 1936, pp. 157–160. On p. 160, five lines from the end, there should not be a minus sign before the symbol c.

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[5] For this distribution, the mean and the mode are identical, so that we take $\theta_0 = 0.6$ years.

ON THE THEORY OF CAPITAL: A REJOINDER TO PROFESSOR KNIGHT

By Nicholas Kaldor

Professor F. H. Knight has done me the honour of writing a detailed reply[1] to a paper of mine[2] containing certain criticisms of his views and published in an earlier number of Econometrica. I do not propose to write a detailed rejoinder to his paper; especially since it contains much with which I agree and the statements with which I do not agree are often so closely interwoven with those with which I do that it would tax the reader's patience too much to attempt to disentangle them. Instead, I shall try to make clear the issues between us, as I see them, by setting out a brief résumé of my own position and comparing it with Professor Knight's. As the reader will observe, apart from a number of minor points, our difference lies in a single major issue.

I

1. The purpose of the Austrian or "time period" theory of capital was to show that "capital" is a distinct factor of production, which can be measured in homogeneous units, both in the production of particular goods and in the economic system as a whole; that the price of this factor is the rate of interest; and that both capital and interest can thus be brought into the framework of production and distribution theory on the same plane as "labour" and "land." (Some economists might, perhaps, disagree with this statement as to the purpose of traditional capital theory. But if this is *not* what the theory was aiming at, what was its purpose?) It rested on two premises. First, the assumption that it is possible to make a "valid general distinction" between capital goods and other productive resources. Second, the attempted demonstration that, with the aid of the concept of the "investment period," the heterogeneous mass of capital goods can be reduced to homogeneity, and thus "capital" can be treated as a quantity per se. Professor Knight rejects both these premises. But since the criticisms on the second count are more numerous, and more difficult to deal with, they may be considered first.

2. It is best to begin by clarifying certain points of methodology. (1) The question whether the "investment period" is something "quantitatively definable" is distinct from the question whether it can also be regarded as a measure of the quantity of capital, as a factor of pro-

[1] "On the Theory of Capital: In Reply to Mr. Kaldor," Econometrica, Vol. 6, Jan., 1938, pp. 63–82.

[2] "Annual Survey of Economic Theory: The Recent Controversy on the Theory of Capital," Econometrica, Vol. 5, Apr., 1937, pp. 201–233.

duction—in other words, the question whether the concept has *meaning* should be kept rigidly separate from the question whether it is *relevant.* Examination of the second question presupposes that the first can be answered in the affirmative. (2) The question whether a definite investment period can be associated with a *single investment*[3] is distinct from the question whether such an "investment period" can be defined for the *economic system as a whole.* It is possible that one of these questions can be answered positively, but not the other. (3) The question whether the investment period can be determined *under stationary conditions* (i.e., in "the stationary state") is distinct from the question whether it can also be determined in the absence of stationariness. Traditional capital theory (both by the Austrians and Wicksell) was elaborated under the postulate of the stationary state; while some of Professor Knight's strictures against the theory[4] clearly arise owing to the absence of stationary conditions. In my view, even if the investment-period theory were found to be a tenable explanation of the nature of capital in the stationary state, it could not be regarded as such for a society which is in a *process of change;*[5] and, this being the case, the question whether the theory is at all applicable for the real world depends on whether the method of *comparative statics* (which treats change as a result, and not as a process) is applicable to problems of capital accumulation. In traditional theory changes in the quantity of "capital" were dealt with merely by comparing different stationary states.[6]

3. In my paper I was first of all anxious to prove that, provided the investment-period concept is quantitatively definable, it is *relevant,* i.e., it will show a correlation with capital quantity. Hence I postulated

[3] A "single investment" is here thought of, not as some concrete capital good, but as something which produces a definite kind of output stream.

[4] Cf. *op. cit.*, p. 67: "One might in theory compute the 'investment period' for a national economy or for the world, but only after the close of its history in either case, or after its entire future history became predictable in quantitative detail." In the stationary state there is no such problem; the history of one day is the same as the history of any other day.

[5] The objections against the "investment period" concept under dynamic conditions cannot be gone into here in detail. It should suffice to repeat what was already stated in my previous paper (cf. *op. cit.*, p. 207) that even if such a concept were definable, it would measure changes in the scale of new investment and not changes in the quantity of capital.

[6] If one takes into account that certain types of equipment are extremely durable, and also indivisible and highly specialised—so that only one or a few units of them are needed—the extreme unreality of this approach in connection with the capital problem becomes at once apparent. It is only for "short-run analysis"—where the amount of existing equipment can be taken as given—that the method of "comparative statics" is at all realistic.

artificial conditions, under which the meaning of the concept was not in question, in order to show that accumulation (=saving) will lead to a lengthening of this period. The reason for this procedure was the belief, gathered from Professor Knight's earlier articles, that his chief objections against the Austrian theory concern the *relevance* of the investment-period concept, and not only its meaning or "reality": at any rate, this is how I interpreted his statement that "the average investment period and the quantity of capital may perfectly well be affected in opposite ways."[7] This statement presupposes that the concept is meaningful; if it has no meaning it is impossible to make any statement about its behaviour. The question of relevance would then not arise.[8] Hence the sections in my paper[9] which Professor Knight prefers "simply to pass over" were devoted to a disproof of the proposition that the "average investment period and the quantity of capital can be affected in opposite ways."

I now realise that I may have been fighting windmills; Professor Knight agrees that the Böhm-Bawerkian theory is valid under the conditions which they postulate, and hence the investment-period concept is not irrelevant, in this sense. He merely insists that the accumulation of capital will not necessarily involve the production of instruments which have a longer construction period, or which last longer, or both. It may do so (or probably will do so) but it may not. This, as I have tried to show in my paper,[10] is perfectly true but not relevant. The average construction period plus the average durability of capital goods merely indicate the average investment period involved in producing *the services of these instruments* and not (or not necessarily) the average investment period of *consumption services*. It is quite possible that the former should be reduced, when the latter is lengthened; when, e.g., capital accumulation implies the introduction of more "automatic" machines, which reduce the amount of "co-operating labour,"

[7] "The Theory of Investment Once More: Mr. Boulding and the Austrians," *Quarterly Journal of Economics*, Vol. 50, Nov., 1935, p. 45.

[8] Even if the theory is found to be wrong because the conditions necessary to validate the investment-period concept do not obtain in the real world, it is important to know whether the things Böhm-Bawerk and his followers were talking about are at all relevant to the problem or not. If they are found to be irrelevant, the whole theory deserves more severe condemnation; it would not even be a "wrong track" in the Jevonian sense, but pure nonsense; and its examination sheer waste of time.

[9] I should like to take this opportunity of pointing out that a vital piece of information was inadvertently left out from the numerical example, introduced in Section II of my paper (p. 210). It was assumed that each of the three types of houses has 75 rooms. Readers who wanted to work through these examples for themselves must have found it very difficult to do so.

[10] *Ibid.*, pp. 226–227.

per unit of output. It is only in cases (such as houses) where the instruments produce consumption services "by themselves," without the aid of co-operating labour, that the two concepts become identical; and, in this case, average durability, or average construction period, or both must become longer, when accumulation takes place.[11]

It is only more recent writers, Professors Machlup and Hayek, who asserted that the accumulation of capital necessarily involves greater "average durability." This of course is wrong; so far as I am aware, neither Böhm-Bawerk nor Wicksell meant to assert it; nor does its denial constitute any sort of disproof of the "Austrian" theory.

4. We can now turn to the more important question whether the concept is meaningful, i.e., whether the investment period is "quantitatively definable."

(i) In the first place, difficulties arise—what has come to be called the "compound-interest problem"—as soon as we take cognisance of the fact that instruments take time to produce and wear out gradually, i.e., that there is no single, definite time lag between "input" and the resultant "output." It is only when growing turnips (which are planted on one day and fully consumed on one day) that the intervening time lag is something entirely unambiguous. In slightly more realistic cases (such as houses which are produced entirely by labour and "free goods" but which take, say, two years to produce and twenty years to wear out) the investment period involves the calculation of an *average* time lag; and this average will be necessarily somewhat arbitrary; it will partly depend on the rate of interest ruling.[12] In my view—and I think this is also Professor Knight's view—this difficulty, taken by itself, would not be so very serious. Although it does make it impossible to determine what will be *the* investment period, it does not make it impossible to give an "index measurement" to it, i.e., to represent its variations by means of an index. And except for the calculation of the

[11] On p. 67 Professor Knight admits that the investment-period theory "can be modified and so stated as to be valid where capital and noncapital agencies co-operate in the creation of final product." On p. 73, however, he says that the only investment period to which he is able to attach meaning is "either some variant of the Jevons-Böhm-Bawerk-Wicksell average construction period and/or average durability, for all the capital items in a system considered individually, or one of these figures computed for the system as a whole, considered as a single investment." The second statement, I think, is inconsistent with the first. If the investment-period concept can also be extended to the case where capital goods co-operate with labour in the creation of the final product, the investment period will be something different from the average construction period and the average durability of instruments; nor will it necessarily vary in the same direction as the latter.

[12] This problem is not new; Wicksell was already well aware of it, cf. *Lectures*, Vol. I, English ed., p. 260. Cf. also my article, *op. cit.*, p. 206.

rate of interest as "the marginal productivity of waiting" which I do *not* consider an essential part of the theory,[13] there is no need for a quantitative measurement of the investment period: an index of "capital intensivity" is all we want.[14]

[13] It is often mistakenly supposed that the entrepreneur, in order to determine his optimum production plan, needs to know the "marginal productivity of waiting" in the Jevonian sense; since, in order to maximise his profits, he must push the application of "capital" up to the point where the rate of interest becomes equal to this marginal productivity (in the same way as the application of "labour" is pushed to the point where the marginal productivity of labour becomes equal to the wage rate). This, of course, is fallacious and is due to a mistaken conception of the nature of capital and interest. The optimum production plan is the one which maximises the rate of return on the investment: this can be determined without any reference to the investment period or its marginal productivity.

[14] As I have shown (*op. cit.*, pp. 212 ff.) such an index is provided (for an "integrated" structure of individual capital goods, where the output stream and the input stream are constant over time) either by the ratio of initial input to annual input, or by the ratio of annual output to annual input (the prices of output units and input units being regarded as given). I also tried to prove that the former index is quantitatively *equal* to the investment period, if compound interest is neglected. In this I was wrong, and the reasoning in footnote 21, p. 213, is erroneous. The mistake arose through the introduction of the equation $C=(b-a)/i$, which of course is a "compound-interest formula" and does not hold when compound interest is neglected. In other words, the result $C/a=t$ was obtained by neglecting compound interest on one side of the equation and "neglecting to neglect it" on the other side. The equation $C/a=t$ will only hold when the rate of interest is zero.

I also misinterpreted Professor Knight's basic equation $a(1+i)^t=b$. Professor Knight was thinking of an investment that is perpetual without maintenance, in which case a, in my terminology, is zero and both indices register infinity. (In his terminology a is the rate of input *during construction*, while b is the rate of output *plus* the value of the regained consumption, a. The value of a in his terminology will not be equal to the value of a in my terminology, except in special cases; while t in his terminology stands for the construction period—the investment period being infinite—in my terminology t is the investment period, and not the accumulation or construction period; these two again only being equal in special cases). Of course, if one assumes a case where the investment is perpetual without maintenance, the investment period necessarily becomes infinite: and, in order to examine the investment-period theory, one certainly should not start by making any such assumption unless one wants to condemn it wholesale at the start, in which case any further analysis of the problem becomes wholly superfluous. And when Professor Knight goes on to say that "neither of these pictures is typical of reality" (p. 701), I leave it to the reader to decide, which of these two cases is more "typical" of reality: the case of investments which are perpetual without maintenance, or of investments which are only perpetual when they are maintained.

I should also like to add that *my* use of the basic equation $a(1+i)^t=b$ does not presuppose that individual capital items are produced by an initial application of factors, extending over a negligible interval, subsequently growing without

(ii) In the second place, there is the further difficulty which Professor Knight brought to light, that maintenance often takes the form of necessary repairs, rather than of replacement; these repairs are a condition for the *functioning* of the equipment itself, so that it is impossible to impute any "investment period" to the input represented by such repairs.

. . . it is incorrect to speak of a time period or degree of roundaboutness unless the capital could be economically disinvested and the flow of final product kept up over the interval measured by this quotient [i.e., the investment period]. This is rarely if ever approximately the case for a single item, and for society as a whole the whole notion is fantastic.[15]

I agree with the latter part of this statement and also with the former part as far as the concept of a "time period" is concerned, but not as far as the "degree of roundaboutness" is concerned. It is impossible to speak of an "investment period" when the maintenance of capital goods is a condition of the current functioning of capital goods. But as I have tried to show, so long as it is still possible to vary the rate of necessary maintenance expenditure, per unit of output, by varying the initial construction expenditure, it is still possible to make production more or less "capitalistic" or "roundabout"; and the *degree of roundaboutness* (measured by the same sort of index as the investment period would be measured by) fulfils exactly the same role in this case as the investment period fulfilled in the previous case. There will still be a correlation, in comparing different stationary states, between the rate of interest ruling and the degree of roundaboutness adopted; and the mechanism described in Section IV in my paper, by which saving, in a barter economy, leads to an increase in the degree of roundaboutness, a lowering of the rate of interest and an increase in the flow of final product, will still be the same mechanism.

I am not sure whether the difference here between Professor Knight and myself is more than a quarrel over words. Professor Knight admits that, *so long as* a distinction can be made between capital goods and other resources, the concept of the "degree of capital intensivity" is valid; and the concepts of the "degree of roundaboutness" and of the

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any further input, and are consumed instantly when "mature" (Knight, footnote 12, p. 71). All that the equation presupposes is that it is possible to "build up" an integrated structure of capital goods (whatever the shape of the input stream or the output stream of individual capital items) which enables the output stream and the input stream, for the *structure as a whole*, to be constant; and that the input stream consists of noncapital services.

Professor Knight agrees that either of these two indices could serve as an "index of capital intensivity" (p. 73), but denies that an index of capital intensivity is also an index to the investment period. On this see pp. 169–170.

[15] Footnote 10, p. 69.

"degree of capital intensivity" are, as far as I can see, exactly the same thing. We both agree, further, that, if capital cannot be "economically disinvested," the concept of an investment period is invalid; no matter how much (or how little) has to be spent on "maintaining" the stock of capital. We appear to differ as to the importance of the notion of the investment period itself, *within the traditional theoretical framework.* The virtue of this concept, in my view, is solely derived from its supposed ability to reduce the existing stock of capital to a homogeneous quantity. If it is impossible to measure capital in terms of an investment period, but *it is* possible to do so in some other way, the investment-period concept goes, and is replaced by this something else, but otherwise the theory remains pretty much as it was. It would be just as true (or more true) to say that the investment period gives, in certain cases, an index to the degree of capital intensity, than to say that the degree of capital intensity, in certain cases, is an index to the investment period. Fundamentally, both these concepts attempt to do no more than to measure the quantity of capital by measuring the ratio of the stock of capital goods to other factors; it is the validity of this "ratio" which is important and not the validity of the "investment period," which is merely one of several ways of measuring it.

Thus in cases where the investment period, though measurable, does not indicate the degree of capital intensity at all, it is the latter concept which is relevant (for the determination of the quantity of capital and the rate of interest) while the former is quite irrelevant. According to Professor Knight, I leave it

> a mystery as to why capitalistic intensivity should be regarded as corresponding in any way with any investment period, or as to what is meant by the degree of roundaboutness for which the ratio is said to be an index. The ratio would apparently have the same meaning in a system in which both machines and slaves lasted forever, and regardless of their origin or what might be known about their past history.[16]

In a society where all capital instruments lasted forever, without maintenance, the investment period will always be zero under stationary conditions—irrespective of the number of such capital instruments. Yet in such a society there will be a productivity rate of interest (the size of the additional consumption stream that can be obtained by the sacrifice of a given amount of current consumption) and what this rate will be will depend on the degree of capital intensivity. (If one wants to *define* capital simply as the investment period, as some Austrian die-hards would, one would have to say that in such a society the amount of capital is always zero in stationary equilibrium; but having said so, one would be no better off than before. Having relegated

[16] P. 73.

the term "capital" to some mystic entity, which has no relevance to economic problems, one would have to turn to some other concept and invent a different name in order to consider the problems of interest, investment, and savings.)

If on the other hand, capital instruments do *not* last forever (which is, after all, the basic assumption on which Austria proceeded and one which—I hope Professor Knight will concede—is not entirely devoid of reality) the index of capital intensivity will always register the same kind of movements as the investment period (provided, of course, that the type of capital goods in existence is not such as to render the measurement of the investment period impossible). Hence where the investment period is a definable concept, it provides a good index to the degree of capital intensivity. (A rather cumbrous index perhaps, for its calculation will by no means be easy in all but the simplest cases.)

(iii) Lastly, there is the "brute fact" emphasised by Knight, that capital goods are not produced by the services of other factors, as is apparently assumed by the Austrians, but that the services of different kinds of capital goods co-operate in producing and reproducing each other. This certainly invalidates the concept of the degree of roundaboutness or capital intensivity when applied to a *single investment*, (i.e., in the production of a single kind of consumption good).[17] But, as I have attempted to demonstrate in Section V, paragraph (iii) and Section VI of my paper—a demonstration which, as far as I can see, was not criticised nor refuted— it would not invalidate the concept for the *system as a whole*, if the latter concept were not deficient *on other grounds*, i.e., on account of the fact that both the services of nonaugmentable resources and consumption services are heterogeneous, and their relative prices are altered by a change in the stock of capital goods.[18] It is only in so far as changes in these relative prices are absent

[17] It does not invalidate the concept of "capital intensivity" for a single firm, or accounting unit, as Professor Knight admits. But it makes it impossible to "lump together" a series of accounting units in such a way that these *together* should only buy noncapital services and only sell consumption services.

[18] Professor Knight affirms the first (the heterogeneity of noncapital resources) and rejects the second (the heterogeneity of consumption services) of these facts as relevant in this connection (footnote 9, p. 69). I confess I do not understand his argument at all. To regard the "quantity of exchange of value of final products as established by perfect competition among sellers and consumers" (footnote 4, p. 64) as given is only possible if the change in question does not affect the *relative* marginal costs of final products which it normally will. If, on the other hand, small changes are contemplated and these consequential changes in relative scarcities are so small that they can be ignored, they can be ignored with the same justification on the factor side as on the product side. In neither case is there any difference. Cf. also note 31 below.

or can be ignored—as for small changes in the stock of capital goods perhaps they can—that we can say how the quantity of capital has been affected, when the amount, or composition, of the stock of capital goods has been changed.[19] For these reasons, it is these latter facts—the heterogeneity of noncapital agencies and of final products—which are the ultimate objections to the traditional view of treating capital as a quantity. This is not to deny the importance of the so-called "circularity argument" (that capital goods produce other capital goods, and so on, in endless succession) but I think the difficulties thereby raised could be surmounted, in viewing a closed system as a whole, if factors other than capital goods were homogeneous in kind and the composition of the final output stream could be considered as given.

II

5. So far we were arguing on the basis that a valid general distinction can be drawn which marks off capital goods from other productive instruments. This, Professor Knight, in the latter part of his article (Section IV), categorically denies; and regards the falsity of this assumption "as the ultimate and crucial fallacy in the time-period theory of capital."[20] To this question we must now turn.

In my article I argued (i) that the distinction between capital goods and other goods is the distinction between augmentable resources and nonaugmentable resources;[21] (ii) that in a society where all resources are augmentable, the rate of interest is uniquely determined from the productive functions, and it is *independent of the extent of accumulation* and is equal to the maximum rate of expansion of the system;[22] (iii) hence the postulate of nonaugmentable resources is necessary in order to explain diminishing returns to capital accumulation.[23]

6. As far as I can make out, Professor Knight would not deny that, in so far as a distinction *can* be drawn, it is the criterion of "augmentability" which is relevant.[24] Nor does he argue that the distinction

[19] Hence the problem of how to determine *when* the quantity of capital remains intact when its composition changes, on account of a change in relative demands for different consumption goods, is only soluble when the change does not affect the relative prices (i.e., when marginal costs are constant).

[20] Knight, *op. cit.*, p. 74.

[21] Kaldor, *op. cit.*, Section IV, p. 218.

[22] *Ibid.*, Section IV, p. 228.

[23] *Ibid.*, p. 231.

[24] The distinction between augmentable or nonaugmentable resources comes close to Böhm-Bawerk's distinction between "original" and "produced" resources, except that it is free from certain implications associated with the latter. In particular, it is not contended that "produced" resources are created from "original" resources, or that "original" resources are necessarily a "gift of nature," and have not been "produced" in some sense, in the past, or even that the

ought to be drawn on some other basis.[25] What Professor Knight denies is simply that such a distinction *can* be drawn; in other words, he denies the existence of nonaugmentable resources. All resources, according to him, are augmentable to a *certain degree;* hence all resources are capital goods.[26]

There can be no doubt that most resources, *as defined and differentiated by the market,* are augmentable to a certain degree. Land can be improved by fertilisation, the supply of skilled labour can be increased by more training, the amount of hydro-electric power can be augmented by the utilisation of yet unexploited waterfalls. Coal available for consumption in large cities can be increased at will by sinking more shafts into the earth and improving transport facilities. But all this is beside the point. Coal in the drawing room is not the same resource as coal in the earth, any more than the house is the same resource as the

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original resources are necessarily physically nonaugmentable, like mineral resources. The quantity of labour is certainly "augmentable," in a physical sense, yet labour will be a nonaugmentable factor if saving does *not* lead to an increase in the available labour supply. The sole criterion is augmentability via capital accumulation.

[25] Professor Hayek has recently adopted a different definition. ("Einleitung zu einer Kapitaltheorie," *Zeitschrift für Nationalökonomie,* Vol. 8, No. 1.) He regards capital as the stock of "nonpermanent goods" or "wasting assets" and he includes only such goods in this category which can be made to yield their services through any space of time (as, e.g., a stock of coal, as against a dwelling house, which lasts a certain number of years even if it is continuously used at maximum capacity; wasting assets are goods with "vorwegnehmbare Erträgnisse"). This definition would include under "capital" such nonaugmentable resources as minerals and would exclude a large part of what is commonly known as "fixed capital." A definition, of course, is a matter of convenience; it all depends on the purpose it is intended for. There can be no doubt that this definition of capital is not relevant for the determination of the productivity rate of interest.

[26] Knight, *op. cit.* pp. 74–78. Similarly he denies that "rent" and "interest" can be treated as different "shares," coming from different sources. "If any fact of economic life is beyond dispute, the fact that the productivity of capital represents the yield of concrete instruments of some sort surely comes in this category. The yield is rent when it is referred to the agency as a quantity of capital, or simply to the capital invested or embodied in it" (p. 74). All "shares," of course, the share of labour not excluded, represent the yield of some concrete agency. The reason for differentiating between rent and interest as distributive shares is the fact that the yield of different kinds of resources is differently affected by changes in the rate of interest. Capital accumulation, if it leads to a reduction of the interest rate, will also lead to a reduction of the net yield (per unit) of those resources which are augmentable; but it will *increase* the net yield of nonaugmentable resources. In a world where the rate of interest is zero the yield of capital goods, viewed as "concrete things," must also be zero; but this surely does not imply that no income will accrue to "land" or no wages to "labour"!

bricks out of which it is made. In all these cases augmentation is only possible at *increasing* cost, and it is only possible up to a point; for in all these cases production embodies an *invariable element*, which cannot be augmented at all. Analytically, at any rate, we must distinguish between hydro-electric plants and mere waterfalls; and it is pertinent to inquire whether more electric power means more plants and more waterfalls or whether it merely means more plants combined with a given number of waterfalls. In the one case the stream of services can be expanded at constant cost; in the other case, at increasing cost.

7. The important question is not so much whether nonaugmentable resources do, in fact, exist or not, but whether diminishing returns *could* exist in the absence of such resources. On this cardinal question Professor Knight returns an unqualified affirmative:

> Mr. Kaldor is (I say) clearly and egregiously wrong in holding that diminishing returns from capital implies changes in proportions between capital as a "factor of production" and (an)other co-ordinate "factor(s)." It is the cornerstone of his argument, and a cardinal error of the whole time-period conception.[27]

Professor Knight does not examine the argument in Section VI of my paper showing that if everything is augmentable, the rate of interest can be derived from the production functions of the different resources and this rate will be independent of the quantity of capital. Instead, he puts forward, as far as I see, three arguments to disprove this proposition.

In the first place he argues that additions to the stock of capital (even if wants and technology are stationary) would never take the same form as units of the previously existing stock.

> In most cases, neither the cost nor the possibility of exact reduplication is in question in determining capital yield or quantity. The reason is simply that *reduplication is not what would happen* [my italics], not the form that capital growth would take, under most circumstances in real life, given perfect freedom of choice—even apart from new inventions or changes in wants. In an extreme case, such as a hydro-electric plant or a railway system, the very notion of physical (Mr. Kaldor says "identical"—p. 219) reduplication is absurd."[28]

It is indeed absurd to assume that by saving one could, or would duplicate the Niagara electricity works or the railway that is alleged

[27] Knight, *op. cit.*, p. 78. Actually I nowhere argued that the change in proportion involved is between "capital as a factor of production" and "other factors" (this statement would have begged all the questions as to nature of capital). What I did argue was that "diminishing returns must always presuppose the existence of some fixed factor as the cause," hence diminishing returns to accumulation must imply a change in the proportion between different types of (concrete) resources. As is obvious from the context, however, this is how Professor Knight in fact interpreted the statement.

[28] Knight, *op. cit.*, p. 78.

to run between Atchison, Topeka, and Santa Fé. But the fact that human ingenuity and thrift are not as yet capable of duplicating such agencies as the Niagara waterfalls or the area known as the United States is surely not irrelevant in this connection. "Reduplication is not what would happen"—but why? If capital accumulation takes the form of creating a resource *B*, and not another unit of an already existing resource *A*, this must imply that *B* is expected to yield more than a second unit of *A*; and since, in accordance with the assumption of diminishing returns, *B* actually yields less than the first unit of *A* has yielded, the yield of the second unit of *A* must be still less than that of the first unit of *A*. If *A* could be expanded at constant cost, the production of *B* would never be resorted to. Or has Professor Knight thrown overboard the assumption that investors want to maximise their pecuniary return?

Professor Knight's second argument seeks for an explanation of these diminishing returns in the realm of consumers' demand.

> It is true that nonreduplicability of existing agencies is a factor in the diminishing returns from investment; but it is a relatively small factor, and operates in different cases in widely different degrees. The main fact lies much deeper, in the nature of products and their "utility," in relation to economic growth . . . [29]

> When the income of an individual increases, in units of fluid purchasing power which he is free to spend as he pleases in a given price situation, he will normally wish only within narrow limits to increase his consumption of products previously purchased. Much more he will wish to add new products to his consumption budget: but again, he will not stop with this, but will to a considerable extent reduce the expenditure on products previously used.[30]

There are two answers to this argument. In the first place, one could argue that these effects will be of the "second order of smalls" and should therefore be ignored in the first approximation. For the increase in income arises *on account of* the accumulation of capital; it will therefore be small *in relation to the investment* and thus the effect of this small increase in income on the productivity of this investment, through its effect on the relative demands for the different products, will be still smaller.[31] Secondly, even if these effects are not negligible, they do

[29] *Ibid.*, p. 80.

[30] *Ibid.*

[31] In footnote 4, p. 64, as already noted, Professor Knight himself argues that the heterogeneity of final products "should be rejected as a factor playing any rôle in the theory of capital"; maintains that in relation to small changes, the exchange value of final products should be taken as given, and criticises me for not doing so. Yet there is no inconsistency in my own position. I was arguing that an increase in capital will affect the relative prices of consumption goods by affecting *relative costs;* while Professor Knight considers the effect on *relative demands*. The effects on relative costs, of course, are of a different order of magnitude from the effects on relative demands.

not prove the *existence of diminishing returns.* In fact the argument could be used equally to show that there will be increasing returns from investment. It all depends on whether the products for which the demand has relatively increased require more or less of the factors which can be created by investment, than the average of all products. In the first case, the marginal rate of return from investment will rise, in the second case it will fall. If these products contain "capital goods" neither more nor less than in the average proportion, the change in relative demands will leave the rate of return unaffected.[32]

But the crux of the whole matter is perhaps found in the third argument:

> Even if increased production took the form of increasing the output of identical goods and services, without change in proportions, and if these were produced by use of the same productive agencies in the same proportions, all agencies being freely augmentable, investment would still be subject to diminishing returns [in the absence of technical improvements] because of the *diminishing utility of total economic income to the individual* [italics mine].[33]

Now whatever may be said as to the previous arguments, there can be no doubt that this last argument is wrong.*The components of the rate of return are *products sacrificed* on the one hand and *products obtained* on the other (both measured in terms of purchasing power, i.e., in terms of one of the products serving as a standard of value); and the diminishing marginal utility of products could just as little affect their value *in terms of products* as a fall in the marginal utility of bread could affect the value of bread in terms of bread. Diminishing marginal utility of total income may be an important factor in determining *the rate of capital accumulation;* but for the determination of the rate of return on investment it is wholly irrelevant.

The proposition that the existence of diminishing returns always presupposes the existence of some "fixed factor" as their cause, and that diminishing returns are entirely a matter of changes in propor-

[32] This is not to deny, of course, the importance of the question raised by Professor Knight that changes in the economic system are "qualitative" and not only "quantitative"; and that it is impossible to regard the number of different goods produced, or even the number of different factors of production as a *datum.* Here I am merely concerned to show that the "qualitative" character of economic changes, however important this may be in a different context, cannot be adduced as an explanation why investment opportunities are limited, i.e., why there are diminishing returns to capital accumulation.

[33] Knight, *op. cit.*, p. 80.

* On reading proof of this *Rejoinder,* Professor Knight asks that notation be made of his agreement that this third argument is wrong. He stands by the conclusion, and the first two arguments, and others which might be given, but his reasoning as quoted is untenable.—EDITOR.

tions of factors, has never been more clearly or persuasively argued than by Professor Knight himself in his earlier writings.[34] It is a proposition on which, ultimately, not only Böhm-Bawerk and the Austrian theory of capital, but our whole inheritance of Ricardian economics, the whole theory of production and distribution, as we know it and teach it, rests. If Professor Knight could convince me that it is wrong, if he could be as persuasive in arguing against the proposition as he was in its favour, I should willingly admit that his recent attack on traditional capital theory had succeeded not merely in eliminating some ill-begotten formulations, but that it had destroyed the whole structure, burying everybody under the ruins. But until a convincing demonstration is forthcoming, I shall remain stubbornly old-fashioned on this point; I shall continue to believe in the Theory of Production, and proclaim the old Knight as against the new!

8. I shall look forward with interest to the new edition of *Risk, Uncertainty and Profit* where that "incubus on economic analysis," the notion of a factor of production, is "summarily eliminated."

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[34] See especially *Risk, Uncertainty and Profit*, pp. 97 ff., and that brilliant essay "Fallacies in the Interpretation of Social Cost," *Quarterly Journal of Economics*· Vol. 38, 1924, reprinted in *The Ethics of Competition*, p. 217.

ON THE MATHEMATICAL HYPOTHESES UNDERLYING CARL SNYDER'S TRADE-CREDIT-RATIO THEOREM*

BY EDWARD V. HUNTINGTON

IN A SERIES of papers beginning about 1924,[1] Dr. Carl Snyder has proposed and defended an important theorem which may be formulated as follows:

In order to preserve the stability of the general price level (P), we must keep the quantity of money (M) proportional to the long-time trend (T') of the volume of trade; that is, we must make $M=kT'$.

His proof of this theorem is based partly on statistical data, and partly on mathematical deductions from the "equation of exchange" associated with the names of Simon Newcomb, Irving Fisher, and others. Since the equation of exchange has been the subject of much debate, and since the steps of Dr. Snyder's mathematical deductions therefrom are not wholly clear, it may be worth while to indicate what the hypotheses are which underlie a logical proof of the theorem. Those economists who do not accept the final theorem will then be able to see clearly the precise points against which their attacks must be directed. The conclusion cannot be overthrown unless at least one of the premises is proved false.

The quantities involved in the equation of exchange are the following:

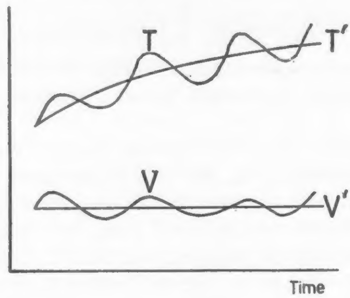
M = quantity of circulating medium (chiefly bank deposits),
V = its "velocity," that is, the ratio of clearings to deposits in a unit time,
P = the average of prices ("general price level"),
T = physical volume of trade (goods exchanged) in the unit time.

Now suppose that V and T are plotted against time, and let V' = the ordinate up to the long-time trend of V ($V=V'+V''$), and T' = the

* This paper has been discussed by Professor Huntington with Mr. Snyder, and meets with the latter's approval.—EDITOR.

[1] "New Measures in the Equation of Exchange," *American Economic Review*, March, 1924; "New Measures of the Relations of Credit and Trade," *Proceedings of the Academy of Political Science*, January, 1930; "Further Measures of the General Price Level and the Trade-Credit Relations for Great Britain," in *La Revue de l'Institut de Statistique*, 2ième Année, Liv. 3, Octobre, 1934; "Deposits Activity as a Measure of Business Activity," *Review of Economic Statistics*, October, 1924; "A New Index of the General Price Level from 1875," *Journal of the American Statistical Association*, June, 1924; "The Problem of Monetary and Economic Stability," *Quarterly Journal of Economics*, February, 1925, resumé with further bibliography.

ordinate up to the long-time trend of T $(T = T' + T'')$, so that V'' and T'' represent the deviations from the trend lines.



Then Dr. Snyder's first hypothesis, based on statistical observations, is that under ordinary circumstances the long-time trend line of V is nearly horizontal; that is,

(1) $$V' = \text{constant}.$$

His second hypothesis, also based on statistical observation, is that under ordinary circumstances the percentage variations in T coincide with the percentage variations in V; that is, in precise notation,

(2) $$T''/T' = V''/V'.$$

It should be noted that this is a direct equation between two pure numbers; no factor of proportionality is involved.

Dr. Snyder's third hypothesis is of theoretical nature, namely, the equation of exchange itself:

(3) $$MV = PT.$$

By direct substitution of (1) and (2) in (3), we obtain at once the formula

(4) $$M/T' = P/V',$$

from which (since V' is constant) the theorem stated above immediately follows; that is, if P is to be kept constant, we must have

(5) $$M = kT',$$

where k is a factor of proportionality.

It may be of interest to note that the second hypothesis may be replaced by a slightly weaker form, namely,

(2a) $$(T/T') = c(V/V')$$

where c is a constant of proportionality. Then from (1), (2a), and (3) we obtain the formula

$$M/T' = c(P/V'), \tag{4a}$$

from which the theorem follows as before.

It is thus seen that the theorem depends solely on the three hypotheses (1), (2) or (2a), and (3). If these three hypotheses are accepted, the theorem itself must be accepted also.

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REPORT OF THE ATLANTIC CITY AND INDIANAPOLIS MEETINGS, DECEMBER 27–30, 1937

THE AMERICAN winter meetings of the Econometric Society were held at Atlantic City, New Jersey on December 27–29, 1937 in connection with the meetings of the other social science societies; and at Indianapolis on December 30, 1937 in connection with the meeting of the American Association for the Advancement of Science.

All the sessions at Atlantic City were held in the Belvedere Room of the Traymore Hotel. The opening session, on Monday afternoon, December 27, was devoted to Theoretical Economics, with Professor Harold Hotelling of Columbia University, president of the Society, in the chair. The following papers were presented:

PAUL A. SAMUELSON, Harvard University: *The Empirical Implications of Utility Analysis.*—Mr. Samuelson stated that it is a widely recognized tenet of modern scientific methodology that, in order for a theory to be meaningful, it must contain empirical implications by which it could be refuted. Surprisingly enough, the operational significance of utility theory has been the object of little investigation in the past. It can be shown that the mere assumption that there exists an *ordinal* preference scale which each individual is conceived to be maximizing subject to given prices and income leads to definite restrictions upon individual and market price-quantity behavior. A method was suggested for the simple derivation of known conditions and new sets of conditions, including restrictions upon the market demand schedules. A brief history of known conditions was outlined.

FRANCIS MCINTYRE, Cowles Commission for Research in Economics and Colorado College: *Price Discrimination Between Two Markets under Imperfect Competition and Monopoly.*—Mr. McIntyre suggested that the word dumping must be divorced from its predatory associations with unfair (i.e., unpopular) competition. To the economist it should merely signify sale for export at a price different from that charged domestic buyers. Such price discrimination has been a regular part of the pricing policy of the U. S. copper industry since 1932 when, following imposition of a high tariff, separate price quotations were established for domestic and export sales. Transportation and tariff barriers protect discrimination in favor of the foreign purchaser, but during the early months of 1937 the direction of dumping was reversed, indicating a high degree of control on the part of the industry over the ultimate disposition of its product. Statistical studies are in progress at the Cowles Commission, comparing the domestic and export demand for U. S. copper as influenced by price level, business activity, etc., at home and abroad. These studies indicate that the demand rose (i.e., the demand curve shifted upward

and to the right) more rapidly for export than for domestic purchases of copper during January–March, 1937, and that, while both demands declined in April, the export demand dropped much further than did the domestic. This coincides, in timing, with domestic dumping during January–March, and a return to export dumping in April, and is offered as additional evidence that charging what the traffic will bear is a part of price policy in this industry.

VICTOR S. VON SZELISKI, Mercer-Allied Corporation: *Advertising as a Variable in the Cournot Monopoly Problem.*—Mr. von Szeliski pointed out that monopoly problems which have been investigated by the mathematical method usually cover only those cases where the monopolist can adjust only price or production. If it is assumed that he can also adjust selling expenses, several new conclusions may be deduced. It can be shown that in general the monopoly price with advertising is always greater than the monopoly price without advertising.

GERHARD TINTNER, Iowa State College: *The Definition and Dynamic Significance of Randomness in Time Series.*—Professor Tintner began with the assumption of an individual who is in the point in time 0 and plans for the period 0, n. He tries to maximize a utility or profit functional $F(0)$, which depends on a subjective factor x, which he can change at will, and an objective factor y, which is independent of his actions but the future course of which he has to forecast. The optimum solution for x, say $\bar{x}(0, t)$, is a functional of these estimated values of y. There are two types of errors possible, where error is defined as any deviation from the optimum solution: errors of the first kind result from the fact that the individual does not actually achieve the set $\bar{x}$, which would maximize the utility functional; errors of the second kind occur if the actual development of y does not coincide with the one anticipated. In both cases the new utility functional which exists after the error has been committed will be different from the first one and the individual will have to find a new set $\bar{x}$ which maximizes the new functional. With errors of the second kind, i.e., errors resulting from faulty forecasts of y, the individual will take into consideration the actual course of y in the past in his effort to get a new estimate of this factor for the future. Hence the new set $\bar{x}$ which is a functional of the new estimates of y will also be a functional of the past y. If we consider the case of the economic system as a whole, the assumption can be made that errors of the first kind are more or less random and hence represent the random or "noncausal" part of the time series. Errors of the second kind enter much more fundamentally and probably have to do with cyclical fluctuations.

At the Monday evening session the general topic was Money and

Credit. Professor Irving Fisher of Yale University presided and the following papers were given:

JAMES HARVEY ROGERS, Yale University: *Government Spending and the Capital Market.*—Professor Rogers as a background for his paper called attention to the credit possibilities of the banking system as shown in charts carried up to May, 1937, and to his paper, "The Absorption of Credit."[1] Government spending may be divided into three categories: (1) pure inflationary spending, (2) noninflationary spending, and (3) deflationary spending. As a first approximation these can be described respectively as deficit spending financed through credit expansion, spending with the national budget in balance, and spending with the national budget running a surplus. Present government spending in the United States is of the third kind, that is, deflationary on net balance. The reason is as follows: At the same time that the Government is turning over to spenders 195 million dollars more than it is taking in simultaneously in the form of taxes (or loans from investors) it is actually taking in an even larger sum than this big deficit in the form of payments under social-security legislation. Moreover, the great bulk of these payments is taken in from very ready spenders and the net deflationary influence is measured by the difference between these social-security contributions and the net budget deficit. This difference is now (November, 1937) estimated for the fiscal year 1937–38 at approximately 370 million dollars. Pursuit of the analysis through a second approximation yields the following conclusions: (1) Not only is the surplus-budget regime proving deflationary, but in addition (2) the reduction of government spending, in that it diminishes the turning of inactive into active deposits, is reducing business activity and incomes, and finally (3) the financial operations of the Treasury in converting short-term obligations held by banks into long-term indebtedness through the investment of social-security funds is proving a deflationary influence of importance. At the present time there are virtually no new flotations in the American capital market. Not only has business not taken up the heavy spending load recently dropped by the Federal Government, but in addition the deflationary influences above described are adding their depressing effects. Hence it is clear that a public-works program of considerable magnitude is needed.

RICHARD DANA SKINNER, Townsend-Skinner and Company: *Suggestions for the Use of Banking Statistics in the Study of Business Cycles.*—Mr. Skinner pointed out that in the past we have used bank figures far too much to analyze the condition of the banks themselves or the general position of the credit structure—and far too little to reveal the

[1] ECONOMETRICA, Vol. 1, Jan., 1933, pp. 63–70.

important economic activities of the community in which we live. In so many ways banking statistics are a symbol or reflection of important human activities that they frequently tell us far more accurately than trade reports the truth about current conditions. Conspicuously in the first half of 1937, a more than seasonal decline of 15 per cent in bank debits in contrast with a rise of some 6.2 per cent in commercial loans indicated that the American public was curtailing its buying activity at the very time when manufacturers were piling up inventories in anticipation of improving sales. This obviously indicated an approaching condition of log jam in the economic system. Yet how few people were interested enough in banking statistics to follow this simple comparison or ratio of spending to borrowing! It was the earliest indication of the depression trend and preceded by nearly eight months the more convincing expression of this new trend in the equity markets and in industrial production figures. Proper correlation of various banking figures can thus show us with surprising frequency beginnings of new trends which sooner or later have their unavoidable impact on short- and long-term interest rates, on the state of trade and earnings, and eventually upon the values of securities based on those earnings.

DICKSON H. LEAVENS, Cowles Commission for Research in Economics: *Five Years of American Silver Policy.*[2]—Mr. Leavens summarized Congressional legislation and administrative action with regard to silver from 1933 to 1937. During this period 1,500,000,000 fine ounces of silver have been acquired; if present gold stocks are not diminished, nearly 1,200,000,000 ounces more must be purchased to carry out the provisions of the Silver Purchase Act of 1934. The American policy drove China off the silver standard and has not encouraged the monetary use of silver in other countries. If silver had been let alone it is probable that, after a period of demoralization not much worse than that experienced by other commodity markets, a fairly satisfactory equilibrium would have been attained. As it is, with China no longer a user, the demand for silver for industrial uses in Western countries and for the ornament trade of India is sufficient to absorb only about half of the current new production, to say nothing of demonetized silver from India and China. The American Treasury is the only buyer for the surplus and the world price is entirely dependent upon its actions.

On Tuesday morning, December 28, the general topic was Welfare Economics. The chairman was Dr. Charles F. Roos of the Mercer-Allied Corporation, and the papers were as follows:

HAROLD HOTELLING, Columbia University: *The General Welfare in*

[2] This paper is published in full under the title of "Five Years of Silver Subsidy," in the *Annalist*, Vol. 51, Jan. 7, 1938, pp. 4–5.

Relation to Problems of Taxation, Railway and Utility Rates.—Professor Hotelling said that while it has not generally been perceived that the problems of taxation and those of railway and utility rate making are closely connected, nevertheless the underlying unity is such that the considerations applicable to taxation are very nearly identical with those involved in proper rate making. He showed that one of the classical theorems, that excise taxes reduce the total national dividend, or "total of satisfactions," by comparison with taxes on incomes, inheritances, and the site value of land, is valid even when full account is taken of relations among commodities. The same argument then shows that the social optimum has as a necessary condition that sales of goods and services shall be at *marginal* cost. The marginal cost of carrying an additional passenger on a railway is usually very close to zero. Thus the conclusion is reached that drastic reductions in railway and utility rates would benefit users of these services to such an extent that they could well afford to pay higher income and inheritance taxes to make good the loss of revenue to the rail and utility enterprises and would still be better satisfied. In the computation of marginal cost it is necessary to include not only the additional labor and materials used for an additional unit of service, but also, when facilities are too limited for the demand, something in the nature of a rental charge for the use of the facilities, in order to discriminate economically among various would-be users. But this rental charge has no fixed relation to the carrying costs of the investment or to overhead charges. A criterion is also established for the decision as to whether new capital investments in such industries should be undertaken. It is shown that ordinary commercial criteria for such investments are too conservative in comparison with that of maximum general welfare.

HAROLD T. DAVIS, Northwestern University and Cowles Commission for Research in Economics: *The Pareto Distribution of Income.*—Professor Davis considered the three questions: (1) What is the frequency function for the total distribution of incomes from the poorest member of society to the most wealthy? (2) Does the distribution appear to be an inevitable one, or may its form be governed by the type of society from which the income is derived? (3) Can any a priori reason be given for the form of the frequency function? The function

$$y=\frac{a}{x^n}\frac{1}{(e^{b/x}-1)}, \qquad n>1,$$

where y is the frequency of those having the income x, measured from the lowest income, was suggested as a possible distribution function. This function yields as an asymptotic approximation the famous

Pareto distribution for large incomes, and has a mode close to zero. The form of the frequency function is derived from the following considerations: Consider a typical income class x, to which N_x individuals aspire to belong. If the total income for that class is I_x, then there will be $P_x = I_x/x$ places in the class to be filled. The number of ways in which P places can be assigned to N individuals is $Q = (N + P)!/(N!\,P!)$. We then introduce the assumption that the rate of change of Q with respect to P varies directly as Q, but inversely as the size of the income, that is, $dQ/dP = bQ/x$. Employing Stirling's formula to evaluate the factorials in Q, and using the assumption just stated, we readily obtain

$$P_x = \frac{N_x}{(e^{b/x} - 1)}.$$

The quantity N_x is determined from the known form of P_x (the law of Pareto) for large values of x. Following a suggestion made by Mr. Carl Snyder at the Research Conference of the Cowles Commission for Research in Economics in 1936, the thesis was advanced that the distribution of incomes as approximately represented by the Pareto law is only one example of a much more general law of inequality, which we might refer to as the law of the distribution of special abilities. Data were given in defense of this proposition to exhibit the Pareto characteristics of the abilities to play billiards, to write mathematical papers, and to produce research in chemistry and physics. The Pareto distribution of executive ability was also shown from the salaries paid executives by corporations.

The Tuesday afternoon session was devoted to Security Valuation and Speculation, with Professor Irving Fisher in the chair. The following papers were presented:

Charles F. Roos, Mercer-Allied Corporation: *Dynamics of the Security Markets.*—Dr. Roos further developed ideas which he had presented at the Research Conference of the Cowles Commission for Research in Economics in July, 1937.[3] Forward discounting of earnings is shown by the fact that for vigorous growing companies the average price-earnings ratios—earnings capitalized in proportion to the rate of interest on AAA bonds—are higher than for stabilized companies and when an upward trend is broken the reaction is sharper than for stable companies suffering a loss in earnings. During the 1925–31 period the earnings ratios have to be adjusted upward by a factor proportional

[3] Cowles Commission for Research in Economics, *Report of Third Annual Research Conference on Economics and Statistics, June 28 to July 23, 1937,* pp. 17–19, 26–31.

to brokers' loans for the account of others which were used primarily to inflate stock prices. Money supply—demand deposits, a portion of time deposits, a portion of brokers' loans plus letters of credit, currency in circulation less float—and its rate of use as measured by building activity are highly correlated with stock prices. A much better correlation is obtained by using bank debits, corrected for financial transactions, wages, and imports, as the monetary series. When the latter series is used no correction needs to be made for the 1925–31 period, presumably because velocity of money used in consumer-goods transactions varies little and the changes in velocity that are important occur in funds used in financial transactions and new building. In mining industries, such as copper, prices of stocks are more influenced by prices of the commodity than by earnings. The speaker then presented a price study for copper which defined the total consumption in terms of (1) automobile production, (2) machine-tool production, (3) utility construction, (4) residential building, (5) industrial building, and (6) general business. Portions of total copper consumption going to each of (1)–(6) were calculated; the sum was total consumption. When price of copper was adjusted by technological cost factors as represented by a smooth downward time trend, the residuals were positively correlated with the total consumption, thus showing that the price phenomenon is associated primarily with supply costs. The demands (1)–(6) were calculated from general factors—in the case of (1) from national income, rate of change of national income, age of cars, and the difference in price indexes of new and used cars, all of which were practically independent of the price of copper. About 90 per cent of the variation in the price of copper was accounted for. Dr. Roos summarized his results by discussing dynamic formulas involving the flow of deposits, the character of bank assets, and political impacts, which he had presented at the Cowles Commission conferences.

A. Andrew Hrubec, Mercer-Allied Corporation: *Speculative vs. Investment Price Behavior of Securities.*—Mr. Hrubec considered in more detail the rate of change in earnings introduced by Dr. Roos[4] as a factor in the mathematical theory of stock prices. When the rate of change is more or less constant in either direction prices will follow regression lines rather closely. When the rate of change increases or declines, sharp fluctuations occur in stock prices with self-generating forces causing overdiscounting and subsequent sharp corrections. This is well illustrated in correlation studies of industrial earnings and other factors against the industrial stock prices.

An aid in timing these moves is afforded in a study of price move-

[4] *Ibid.*

ments of suitable price indexes of speculative and investment stock prices along with indexes of volume of trading in each group. Group indexes were constructed of some thirty speculative issues and a like number of investment issues. Care was taken to include various industries and to match the stocks in the investment group with the stocks of similar types of companies in the speculative group. The volume index of each group is a combined index of the ratios of the volume of each individual company to a suitable average of the daily volume for a period immediately preceding. A further aid in interpretation is afforded by plotting the divergence of the speculative price index from the investment price index. Conclusions can only be briefly stated here. In general, low-priced issues are the first to reach highs as well as lows and higher-priced issues touched their highs and lows after the low-priced speculative issues. After a decline an oversold situation is indicated by a sideways movement in the speculative issues while investment stock prices start to climb. In the later part of a recovery speculative stocks move very rapidly until an overbought situation occurs in the market, indicated by a leveling off in speculative prices. At the tops volume activity is mostly in speculative stocks and the end of a decline in the market is not indicated until the investment stocks definitely assume leadership in trading. The differential movement of the two groups is closely linked to changes in the rate of change in earnings and, though subject to a great deal of interpretation, a study of both seems to give the timing of significant moves in the market.

NIKITA ROODKOWSKY, Columbia University: *Factors Influencing Bond Yields.*—Dr. Roodkowsky discussed conclusions reached by statistical analysis of railroad financial tests consisting of items from financial statements and ratios of these items. There were two problems to solve: First it was necessary to determine whether or not a given test differentiates between financially good and poor railroads; Second, to investigate the possibility of establishing permanent standards based upon financial tests. To find an answer to the above problems correlation coefficients were computed between each of thirty-nine most-used financial and technical tests calculated for all class A-1 railroads and the bond yields of these railroads. The following railroad tests were found to correlate significantly with bond yields: Net Income to Total Income; Gross Operating Revenue to Fixed Charges; Net Income; Net Income to Fixed Charges; Net Ton-miles per Train-hour; Freight Density. By applying mathematical probabilities to railroad tests used by several states as legal requirements for railroad bonds it was definitely established that the following legal tests do not differentiate between good and defaulted railroads: Mileage; Gross Earnings; Cash Dividends to Capital Stock; and Total Debt to Capital

Stock. Although several railroad tests were found to differentiate between railroads in good and poor financial position, the results of application of tests of significance and theory of variance to railroad financial data arouse a doubt as to the possibility of obtaining permanent standards for railroad bonds.

The session on Wednesday morning, December 29, was a Cournot Memorial program in celebration of the centennial of the publication of *Recherches sur les principes mathématiques de la théorie des richesses* in 1838. Professor Harold Hotelling presided and the following papers were given:

A. J. Nichol, University of California: *Tragedies in the Life of Cournot.*—Professor Nichol remarked that tragedies in the lives of economists are usually of a mental rather than a physical nature. So it was in the main with Cournot. Yet at the root of his disappointments there was a physical infirmity. As a child he read voraciously in spite of a premonition that he would ruin his eyesight. This premonition was all too completely fulfilled. The errors in the original edition of his great work in economics, conveying the appearance of gross carelessness, were really the result of partial blindness. A few years after the publication of his *Researches* Cournot, though still a comparatively young man, was forced because of his eyes to renounce forever all work of a distinctly mathematical character. Except as he suffered from eye trouble, Cournot was never in any very prolonged physical distress. For almost thirty years he capably fulfilled various administrative duties in the public school system of France. Yet his life was not a particularly happy one. Death took away from him an only son of great promise. From the viewpoint of the progress of economic ideas the great tragedy of Cournot was that, being almost blind, he was unable to push further the distinctive work which he began, but which others have had to carry on.

Irving Fisher, Yale University: *Cournot Forty Years Ago.*[5]—Professor Fisher recalled that mathematical economics, now so generally recognized, was still looked at askance in 1897 when the English edition of Cournot appeared. This had been true for many years. Ingram, in his *History of Political Economy*, 1888, had said of mathematical economics (p. 182), "there is, therefore, no future for this kind of study; and it is only waste of intellectual power to pursue it." Nevertheless it was even then showing signs of a healthy growth. As was stated in Professor Fisher's article on Cournot in the *Quarterly Journal of Economics* for January, 1898:

> For good or for ill the mathematical method has finally taken root, and is flourishing with a vigor of which both its friends and enemies little dreamed. Sixty years ago the mathematical treatise of Cournot was passed over in silence, if

[5] This paper will be published in full in a later issue of Econometrica.

not contempt. To-day the equally mathematical work of Pareto is received with almost universal praise. In Cournot's time "mathematical economists" could be counted on one's fingers, or even thumbs. To-day they muster some thirty active enthusiasts and a much larger number of followers and sympathizers. In 1838 there seems to have been no institution of learning besides the Academy at Grenoble, of which Cournot was rector, where "mathematical economics" were employed or approved. In 1898 there are at least a dozen such institutions, and in England alone half that number, Oxford and Cambridge among them. It is in France, the prophet's own country, where he is still without honor.

During the last forty years both the appreciation of Cournot and the appreciation of mathematical economics in general have grown amazingly and especially in the last five years since the Econometric Society was founded.

In the discussion Professor WILLIAM JAFFÉ of Northwestern University, invoking a phantasy of Cournot in heaven, discussed Cournot's influence on Leon Walras and Alfred Marshall. Walras was the first economist to have acknowledged indebtedness to Cournot. In fact there are more striking similarities in form and content between the work of these two French economists than the general acknowledgments would indicate. Dr. Lilly Hecht showed in her *A. Cournot und L. Walras, ein formaler und materialer Vergleich wirtschaftstheoretischer Abteilungen* that, although the value concepts of both authors are only formally identical, their theories of general equilibrium in exchange are, to a large extent, identical in substance as well as in form. Alfred Marshall is another giant of late nineteenth-century economics who built on a foundation laid by Cournot. Miss Isabelle McCormick (Mrs. John Wadsworth) in a master's thesis written under Professor Jaffé's direction at Northwestern University elaborated on the similarities between (1) Cournot's and Marshall's empirical demand functions; (2) their categories of cost functions; and (3) their uses of taxes and bounties as analytical devices. Marshall was certainly mistaken in his charge that Cournot did not perceive the monopolistic implications of a decreasing cost function. May the shade of Cournot find consolation for the neglect he suffered here below in the appreciations which mortals can now belatedly bestow on his century-old *Researches into the Mathematical Principles of the Theory of Wealth!*

The topic of the final session on Wednesday afternoon was Industrial Economics. The Chairman was Professor Harold Hotelling and the papers were as follows:

CHARLES F. ROOS and VICTOR S. VON SZELISKI, Mercer-Allied Corporation: *Economic Balance and Its Relation to Production and Pricing Problems.*—The speakers stated that in practical economic analysis the function of the econometrist is to fill in as much as possible of the treacherous area between fact and conclusion with exact calculation. Business action and policy decisions can usually be reached only

after the exercise of considerable judgment and inference, which do not follow immediately or unequivocally from the admitted facts. The dangerous area between facts and decision is where the practicing economist can easily go astray. The filling-in can be done (a) by splitting up complex facts into more simple facts, and (b) by combining several facts into single facts. As examples of the latter type of construction, several PQ lines were exhibited (price$\times$volume$=$PQ). Partial correlations were shown to illustrate how the Federal Reserve Board Index and price indexes of finished goods and of raw materials can be combined to give value of products in manufacturing and cost of raw materials. Wages were estimated from the Bureau of Labor Statistics pay-roll index. By subtracting wages and cost of raw materials from value of product a figure representing overhead plus gross profits in manufacturing is reached. This figure, which can be kept up to date weekly, furnishes a very sensitive and significant datum for business-cycle study; expansion in employment occurs when it is increasing and contraction when it is decreasing. Other PQ lines were shown for the building industry and for the operations of the U. S. Steel Corporation. In the former case a study by the authors of St. Louis was extended to urban building in the United States. Relationships similar to those obtained in the St. Louis study were exhibited except that the lag was shortened to one year. The principal factor changing the need for residential building was shown to be the farm-city movement of population.

Wassily Leontief, Harvard University: *Empirical Application of the Economic Theory of General Interdependence.*—Professor Leontief defined his task as an attempt to fill the "empty boxes" of the economic theory of general interdependence. He described the layout and principal results of a study aimed at empirical determination of functional interrelations between the main "technical" parameters of the economic system of the United States and the values of its variables, i.e., prices and outputs of various commodities and services.[6] The underlying theoretical scheme was characterized as that of a simplified Walrasian system. The speaker stressed the difference between his approach and the other similar attempts to empirical application of the theoretical concept of general interdependence. The majority of investigators in this field are constrained to reduce the number of the variables to a few aggregative price and quantity indices because they

[6] A detailed presentation of the statistical background of this investigation can be found in his article in the *Review of Economic Statistics*, August, 1936, on "Quantitative Input and Output Relations in the Economic System of the United States." A complete description of the theoretical setup as well as the analysis of the empirical results obtained was published in the same *Review* in August, 1937, under the title of "Interrelation of Prices, Output, Savings and Investment."

base their analysis on the small number of rather complicated non-linear equations. He himself, on the contrary, reduces the theoretical issue to a solution of large systems of relatively simple linear equations and thus is able to operate with a much greater number of variables. The computation of hundreds of ten-row determinants involved in the numerical solution of such systems was made possible through the use of the (recently invented) Simultaneous Calculator.

AUGUST LOESCH, Rockefeller Fellow: *The Nature of Economic Regions.*—Dr. Loesch stated that a theoretical analysis starting from the assumption of a uniform plain shows three types of economic areas: (1) hexagonal market areas surrounding every center of production or consumption; (2) a net of such areas for every commodity; (3) a systematic arrangement of the various nets, based mainly upon the advantages of local concentration of production, consumption, or trade, and upon the most economical layout of lines of communication. This self-sufficient system represents the ideal economic region. The ideal pattern is considerably modified by geographical and other spacial inequalities of importance in the real world. In particular, the nets or their centers are often compressed on a relatively small space (belts or districts). Nevertheless, beneath a sphere of irregular market areas we find a regional substructure almost everywhere. It differs from the ideal pattern inasmuch as real regions are not self-sufficient. But they are at least more so than either markets, or districts and belts. On the other hand, due to their more complex nature they are much more exposed to disturbances. Such disturbances are not of equal strength everywhere. Consequently the importance of the regional substructure varies, and it is not very useful to try to divide a state up into its regions. With respect to some parts of it, however, the regional concept may be most realistic.

* * *

The program of the Econometric Society at Indianapolis consisted of a single session held jointly with the American Mathematical Society, the Institute of Mathematical Statistics, the Mathematical Association of America, and Sections A (Mathematics) and K (Economics, Sociology, and Statistics) of the American Association for the Advancement of Science. This was held in Jordan Hall of Butler University in the afternoon of Thursday, December 30. The following paper was presented:

GRIFFITH C. EVANS, University of California, Address as Retiring Vice-President of Section A: *Recent Mathematical Progress in Theoretical Economics.*—Professor Evans compared the situation at the present day with that at the time, now ten years past, when he still dared to call himself, with Schultz, Hotelling, and Irving Fisher, a pioneer in the

subject in this country. At the present time a number of competent analysts are using a wealth of powerful methods for publications in various journals and presentation in conferences such as those of the Cowles Commission in Colorado Springs. The mathematical setup consists in the treatment of data—the data of statistics—with relation to problems. It is important to realize that the purpose of statistics is merely to answer questions and to make decisions between various formulated possibilities. Statistics themselves have no brains or imagination, and, strange as it may seem to some, do not propose theories on their own account. The problems are of two kinds, those that refer to a particular portion of the economic system, isolated for the sake of the treatment, like the theory of monopoly, of security prices, of depreciation, etc., and those that refer to the movement of the system more or less as a whole—problems in the large, or "macrodynamic" problems, if one can stand the word, such as the theory of price levels, of money, of unemployment, and of rate of interest. One source of such problems is the attempt to rationalize the suggestiveness of Keynes's recent discussions of money. In general the methods of the two kinds of problems are those of the calculus of variations. with the integral often replaced, where there is a typical time lag, by equations involving differences. But no mathematical method is valid unless it yields results which vary slightly when data vary by small amounts, since economic data are of course only approximate. This is not a universal property of mathematical methods. The speaker illustrated his remarks by a discussion of an unpublished paper of J. M. Thompson on the difference between a broker's buying and selling prices of a farm commodity (problem of type 1) and one of Kalecki's on the theory of business cycles[7] (problem of type 2), which investigates the periodic solutions of a certain differential-difference equation.

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Research in Economics
Colorado Springs, Colorado

[7] "A Macrodynamic Theory of Business Cycles," Econometrica, Vol. 3, July, 1935, pp. 327–344.

ANNOUNCEMENT OF THE CRACOW MEETING SEPTEMBER 19–23, 1938

The Eighth European Meeting of the Econometric Society will be held in Poland at Cracow, September 19–23, 1938. The date has been chosen so as to follow the meeting of the International Institute of Statistics at Praha (Prague). A full announcement will follow in the July issue of Econometrica.

Wladyslaw Zawadzki